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Coproduct Technologies: Product Line Design and Process Innovation

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The simultaneous production of different outputs (coproducts) is observed in the chemical, material, mineral, and semiconductor industries among others. Often, as with microprocessors, the outputs differ in quality in the vertical sense and firms classify the output into different grades (products). We analyze product line design and production for a firm operating a vertical coproduct technology. We examine how the product line and profit are influenced by the production cost and output distribution of the technology. We prove that production cost influences product line design in a fundamentally different manner for coproduct technologies than for uniproduct technologies where the firm can produce products independently. For example, with coproducts, the size and length of the product line can both increase in the production cost. Contrary to the oft-held view that variability is bad, we prove the firm benefits from a more variable output distribution if the production or classification cost is low enough.

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1. Introduction

Coproduct production, whereby different outputs are simultaneously produced in a single run, is a fundamental attribute of the process technology in a vast array of industries. It is pervasive in many segments of the agricultural, chemical, materials, minerals, and semiconductor industries among others. Oftentimes, such as the joint production of acetone and phenol by the cumene process, the outputs have distinct end uses, that is, they differ in a horizontal sense. In many other cases, semiconductors, for example, unavoidable variations in the inputs or processing environment lead to outputs with the same basic purpose but that vary along a dimension for which customers have a vertical preference ("more is better")—speed for microprocessors, luminescence for light-emitting diodes (LEDs), and efficiency for photovoltaic (PV) wafers. Many important industrial products, such as abrasives, coatings, pigments and pharmaceutical excipients, are produced and sold as powders. Vertically differentiated coproducts arise in these industries because particle characteristics, such as shape and size, can heavily influence a product's performance but can be difficult to control; for example, the process technology for industrial diamonds creates crystals of varying shapes, and shape is a key determinant of impact strength. Hereafter, we will use the terms *horizontal* or *vertical* to distinguish between coproduct technologies when necessary.

Classification, the sorting of an output stream by quality, is an important marketing and operations strategy for firms reliant on vertical coproduct technologies. In its simplest form (which we will call "separation"), a firm separates the output into two streams and sells only the stream that meets some specified quality threshold, for example, maximum particle size for ultrafine nickel powder (JFE Mineral Company 2005). A more sophisticated version involves splitting the output into multiple quality grades as is done for microprocessors, LEDs, PV wafers, liquid crystal displays, and industrial diamonds; this is called "binning" in the semiconductor industry, with different bins referring to different grades. Classification is predicated on the willingness of customers to pay higher prices for higher-quality products and on the ability of the firm to sort its output by quality. This highlights the interdependence of marketing, operations, and process development in coproduct firms.

Product line design (choosing how many and what quality grades to offer and their associated prices) must reflect process characteristics (the output distribution) and operations capabilities (production and classification costs) in addition to customers' quality valuations. Production decisions (what quantity to produce) cannot be separated from product line design or process characteristics. Process innovation (designing a "better" process) cannot be evaluated in isolation of the firm's classification strategy.

In this paper we analytically examine the product line design and production decisions of a monopoly firm that operates a vertical coproduct technology. We assume that each customer represents a very small fraction of the firm's overall demand, and so the customer base can be represented as a multitude of infinitesimal entities. We explore how the firm's product line and profit are influenced by the cost and output characteristics of the process technology. The extant literature on product line design (reviewed below) implicitly adopts a "uniproduct" technology paradigm in which the firm can produce each product independently; this is not possible with coproducts because the relative quantity produced of each product depends on the technology's output distribution and the firm's product line design choice. We show that quality availability, that is, the constraint on relative supplies, replaces costliness of quality as a fundamental driver of product line design. Furthermore, this leads to diametrically opposed findings to those in the uni-product technology literature. For example, different from Netessine and Taylor (2007), we show that the size and length of the product line, that is, the number of products offered and the difference in quality between highest and lowest quality products, can both increase in the marginal production cost rather than decrease. They always increase if the classification cost does not depend on the number of grades, but can decrease otherwise. Contrary to the oft-held view in process improvement that variability reduction is desirable, we prove that the firm can strictly benefit from a more variable output distribution (in the mean-preserving spread sense) if its production or classification cost is low enough. This implies that a process innovation that leads to a lower-mean/higher-variance process can be a strict improvement. We show that the capability to classify into multiple grades (rather than separating into one grade) is particularly important if production costs are high.

The remainder of this paper is organized as follows. Section 2 reviews relevant literature. Section 3 describes the model. Section 4 analyzes the pricing, grade specification, and production decisions. Section 5 explores the influence of the process technology (cost and output distribution) on profits and product line design. Section 6 extends our results to allow for randomness in the underlying output distribution. Section 7 concludes. All proofs are given in the appendix, §A.3.

2. Literature Review

Despite its prevalence in practice, coproducts have received surprisingly little attention in the operations literature. Taking the product line and prices as given, which implies product demands are exogenous, the vertical coproduct literature has traditionally focused on production and downward substitution whereby demand for a lower-grade product can be satisfied by a higher-grade product at the lower-grade price (Bitran and Dasu 1992, Bitran and Leong 1992, Bitran and Gilbert 1994, Carmon and Nahmias 1994, Gerchak et al. 1996, Nahmias and Moinzadeh 1997, Hsu and Bassok 1999, Rao et al. 2004, Ng et al. 2012). This substitution option provides flexibility that is valuable in the presence of demand or grade-proportion uncertainty. Bansal and Transchel (2011) extend this literature by allowing for customer-driven substitution in a two-product model, whereby an exogenous fraction of low-grade customers will buy the high grade if the low grade is unavailable for tactical or strategic reasons. Downward substitution, sometimes called upgrading, has also been studied in uniproduct settings, for example, Netessine et al. (2002) and Shumsky and Zhang (2009).

Recently, Boyabatli et al. (2011), Boyabatli (2011), and Boyabatli and Nguyen (2011) explored risk management through integrated procurement and production decisions in the context of agriculture industries with coproduct technologies, for example, beef, cocoa, palm, sugar, and their derivative products. These contexts exhibit significant uncertainties in demand and/or input prices, and there exists spot markets and/or forward contracts for input purchase and/or output delivery. Focusing on the case of two products, these papers explore procurement contract design (Boyabatli et al. 2011, Boyabatli 2011) and capacity investment (Boyabatli and Nguyen 2011) in vertical (Boyabatli et al. 2011) or horizontal (Boyabatli 2011, Boyabatli and Nguyen 2011) coproduct technologies. Motivated by an oil-refinery context, Dong et al. (2012) examine procurement, processing and blending decisions in a two-product horizontal coproduct model with spot markets and intermediate-product conversion flexibility. The contracting, capacity-portfolio, and/or flexibility-investment focus of these agricultural-product- and oil-refinery-motivated papers is very different from our focus on product line design, production, and process characteristics.

None of the above coproduct papers consider pricing or product line design decisions, and other than

Boyabatli et al. (2011), which assumes price-sensitive demand and a market-clearing strategy, prices do not vary with the quantity sold. Tomlin and Wang (2008) explore the role of pricing and substitution in a two-product vertical coproduct system where the product qualities are given and quality-sensitive customers make purchase decisions in a utility-maximizing fashion. Min and Oren (1996) examine optimal allocation rules in a quite general vertical coproduct model with utility-maximizing customers, but the production quantity is fixed. Their formulation allows for the possibility that the firm can choose the quality level for each grade, but this possibility is only briefly treated by way of a numerical example that examines pricing in a three-product instance. Our paper extends the coproduct literature by exploring the product line design challenge and doing so in a manner that is integrated with the production decision and process characteristics. In a recently completed working paper, Deb et al. (2012) explore product line design in an exogenous-price coproduct setting.

The study of product variety has a rich history in the economics and marketing literatures for both horizontal and vertical differentiation; see Lancaster (1990) for a review. In the vertical-differentiation setting, Mussa and Rosen (1978) and Moorthy (1984) deserve particular mention due to their consideration of utility-maximizing, quality-sensitive customers in product line design, an approach we adopt in this paper. The operations literature has devoted significant attention to managing product variety through such strategies as quick response and delayed differentiation; see Tayur et al. (1999) and references therein. It has also examined how operations considerations influence product line design in vertical (Netessine and Taylor 2007) and horizonal (Mendelson and Parlaktürk 2008, Alptekinoğlu and Corbett 2008) settings. The related problem of product assortment, that is, the selection of what products to stock from a predetermined set, has been extensively examined in the operations literature, for example, Kök et al. (2009), Tang and Yin (2010), Pan and Honhon (2012), and references therein.

The product line literature has implicitly adopted a uniproduct technology paradigm that dominates the operations literature; that is, the production technology allows the firm to produce each product independently so that the quantity of one product need not have any relation to the quantity of another unless there is a common capacity constraint.[1] A tension arises in product line design because customers value quality (to different degree), but this quality is costly, that is, the marginal production cost increases in a product's quality. The production-quantity independence of uniproduct technologies does not hold for coproduct technologies (even in the absence of capacity constraints) because the firm makes a single quantity decision that translates to quantities of various outputs in proportions that depend on the product line design. This proportionality dependence renders product line design for coproduct technologies fundamentally different because proportionality leads to supply-constrained product line design and because the choice of qualities influences the product supplies.[2] As we show later, this endogenous quality availability replaces the costliness of quality as a fundamental driver of product line design for coproduct technologies. This crucial distinction leads to some directly opposite findings to those in the uniproduct papers. When there are no fixed costs to adding products to the line, it has been shown that (i) the optimal product line is independent of the customer-type distribution (Pan and Honhon 2012, Corollary 4, p. 262), (ii) the firm offers only one product (the highest quality one) if the marginal production cost is independent of quality (Bhargava and Choudhary 2001, Theorem 1, p. 96), and (iii) the size and length of the product line both decrease in the marginal production cost (Netessine and Taylor 2007, Result 1, p. 109 and Result 6, p. 112).[3] We prove that none of these results hold in our coproduct setting.

Our paper is somewhat related to the literature on process improvement and innovation. Oftentimes, process improvements are assumed to reduce production costs in the economics (e.g., Spence 1984, Lambertini and Orsini 2000), marketing (e.g., Gupta and Loulou 1998), operations (e.g., Fine 1986, Gilbert

[1] Deneckere and McAfee (1996) provide an interesting extension to this uniproduct technology paradigm. They model a two-product firm (high and low quality) with a uniproduct technology for the high-quality product (i.e., no direct coproducts), but the firm can purposely damage the high-quality product to create the inferior one. The supply of the low-quality product is constrained by the production quantity of the high-quality one in this case. Damaging allows the firm to discriminate between customers of different valuations by offering products of different quality, and this can benefit the firm and the customers.

[2] In a uniproduct setting, Dana and Yahalom (2008) explore a resource-constrained version of the model in Mussa and Rosen (1978), but the resource constraint is exogenously given (as opposed to endogenous as in our coproduct model), and its aggregate nature does not capture individual product inventory constraints that arise in multiproduct settings.

[3] More generally, if the marginal production cost $c(x)$ as a function of product quality x is given by $c(x) = cz(x)$, then the results in Bhargava and Choudhary (2001) establish that the optimal product line is independent of c, with the product line comprising all available products if $z(x)$ is convex but only the highest-quality product if $z(x)$ is concave. Netessine and Taylor (2007) adopt a marginal production cost $c(x) = cx^2$ and refer to c as the costliness of quality in Results 1 and 6. Clearly the marginal production cost $c(x)$ decreases in c, and so we describe their result in terms of marginal production cost rather than costliness of quality.

et al. 2006), and strategy (e.g., Adner and Levinthal 2001) literatures. Other times, process innovations increase capacity or yield; see, for example, Porteus (1986), Pisano (1996), Hatch and Mowery (1998), Terwiesch and Bohn (2001), Wang et al. (2010), and references therein. In addition to production cost, we analyze how the output distribution influences a firm's profit and product line to help answer the question of what constitutes process improvement in coproduct technologies.

In closing, we note that the recent focus on sustainability in operations has brought attention to coproduct technologies in relation to emissions accounting (Keskin and Plambeck 2011) and by-product synergies (Lee 2012), but environmental considerations and opportunities are not the focus of our paper.

3. The Model

As with much of the coproduct literature, for example, Gerchak et al. (1996), Hsu and Bassok (1999), Rao et al. (2004), Tomlin and Wang (2008), and Boyabatli et al. (2011), we consider a single-period model. We next describe the production, product line, classification, and customer elements of the model. We then conclude by summarizing the firm's decisions.

3.1. Production

The firm operates a coproduct technology whose output varies along a single attribute x, for example, speed of microprocessors, for which "more is better"; that is, each customer unambiguously enjoys a higher gross utility when offered a higher quality. Single-attribute vertical differentiation is commonly adopted in the literature on product line design; see Mussa and Rosen (1978), and Netessine and Taylor (2007).[4] We use the distribution $F(x)$ to represent the output quality spectrum for a production batch; $F(x)$ is increasing and continuous, and $\bar{F}(x) = 1 - F(x)$. We denote $f(x)$ as its density function and $[\underline{x}, \bar{x}]$ as the support, where $0 < \underline{x} < \bar{x}$. For any given interval $[x_i, x_j] \subseteq [\underline{x}, \bar{x}]$, $\int_{x_i}^{x_j} f(x)\,dx$ represents the *proportion* of outputs whose quality levels lie between x_i and x_j, and $Q\int_{x_i}^{x_j} f(x)\,dx$ represents the *amount*, where Q is the production quantity. We assume that the distribution $F(x)$ is deterministic, but we extend our results to the stochastic output distribution case in §6. The production cost $C_P(Q)$ is assumed to be (weakly) convex in the quantity with $C_P'(Q) > 0$. We assume unsold material has no salvage value or disposal cost.

[4] Multiple attributes may be relevant for certain products, for example, luminescence (brightness) and chromaticity (color) for LEDs. A single-attribute model can be viewed as choosing the product line design for a certain value of the other attribute, for example, how to design the luminescence grades for a particular chromaticity.

3.2. Product Line

A product line is specified by the set of grades (or "bins") that the firm makes available to customers; that is, a product line (with N grades) is defined by the vector $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, where x_n is (weakly) increasing in $n = 1, \ldots, N$. For ease of notation, define $x_{N+1} = \bar{x}$. Grade n is the interval $[x_n, x_{n+1})$. Outputs with quality levels in $[\underline{x}, x_1)$ are abandoned. This is without loss of generality as the firm can always set $x_1 = \underline{x}$ when designing the product line. The quantity of grade n is $Q_n = Q[F(x_{n+1}) - F(x_n)]$. We define the echelon quantity for grade $n = 1, \ldots, N$ as the total quantity of grades $n, n+1, \ldots, N$, and so $Q_n^E = Q\bar{F}(x_n)$. The number of grades N and their specification $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ are set by the firm.

3.3. Classification

Classifying the output requires the ability to test and sort each unit by the quality attribute. In semiconductor contexts this is done by a machine that tests each chip and then places it in the appropriate bin. For powder products that are classified by particle size, test and sorting is carried out simultaneously by passing the batch of powder through a vibrating machine with multiple sieves whose mesh sizes correspond to the grade specifications. Classification incurs a cost that increases in the production quantity Q (because the output Q has to be classified) and the number of grades N the output is sorted into. We assume the classification (or "binning") cost is separable in the quantity and the number of grades, and adopt the cost structure $C_B(Q, N) = b_0 + b_1 Q + b_2(N-1)$, where b_0 represents the fixed or set up cost associated with operating the classification technology, b_1 represents the quantity-related marginal classification cost, and b_2 represents marginal cost associated with grades, for example, the increased processing cost due to having an additional sieve in the powder classification context.

3.4. Customers

We assume a deterministic population size, scaled to one without loss of generality. Deterministic demand is a common assumption in coproduct papers, for example, Bitran and Gilbert (1994), Gerchak et al. (1996), and Nahmias and Moinzadeh (1997), and product line design papers, for example, Netessine and Taylor (2007) and Pan and Honhon (2012). Customers are infinitesimal and vary in their valuation of quality. We use $\theta \in [\underline{\theta}, \bar{\theta}]$ to denote a customer's marginal willingness to pay for quality. Thus, upon receiving a product with quality x, her gross utility is θx. We assume that the manufacturer cannot directly observe the customers' preferences; thus, the willingness to pay also corresponds to the customer's "type." Each customer obtains a null (zero) utility if she walks away empty handed, irrespective of her type. Customer heterogeneity is captured by a distribution function $G(\theta)$, with $g(\theta)$ being its corresponding density and $\bar{G}(\theta) = 1 - G(\theta)$. We assume that $G(\cdot)$ is an increasing failure rate (IFR) distribution (Lariviere 2006).

Although each grade reflects a range of quality, for example, grade n is the quality interval $[x_n, x_{n+1})$, we assume that customers assign the lowest quality level when evaluating a grade, for example, customers treat grade n as having a quality of x_n. In effect, customers either do not care or ignore the possibility that they might get a quality higher than x_n when receiving grade n. This reflects contexts where customers base their valuation of quality (of a given grade) on its worst-case quality. This is the case with microprocessors, for example, because customers, for example, laptop manufacturers, assemble the microprocessor with other components and can only guarantee to their customers that the processor speed exceeds some particular level. Our results can be readily adapted to cases where customers assign the highest quality level when evaluating a grade. In addition to being grounded in reality, this single-point evaluation assumption allows us to bypass the complicated customers' belief formation process. Suppose, on the contrary, that a customer evaluates a grade by the average quality within the grade. In such a scenario, to form the correct (rational) expectation, each customer needs to accurately estimate the manufacturers' production distribution $F(x)$. This is conceptually feasible but hardly achievable in practice, because the output distribution is not known by customers.

Customers make their purchasing decisions simultaneously. Since the supply of each grade is limited, some customers may not obtain their desired products if the total number of requests for a grade exceeds its supply. In the event that demand exceeds supply we assume that the firm may use downward substitution and/or customers may spill down to their next-preferred lower-quality grade. In fact (as discussed in the proof of Proposition 1), all our results hold even if substitution and/or spill-down are not allowed.

3.5. The Firm's Decisions

The firm chooses a production quantity Q, grade specification vector $\mathbf{x}$ (which includes the number of grades N), and price vector $\mathbf{p}$ to maximize its profit $\Pi(Q, \mathbf{x}, \mathbf{p}) = R(Q, \mathbf{x}, \mathbf{p}) - C_P(Q) - C_B(Q, N)$, where $R(Q, \mathbf{x}, \mathbf{p})$ is the revenue and $C_P(Q)$ and $C_B(Q, N)$ are the production and classification ("binning") costs, respectively. Because there is no uncertainty in the base model, the decision sequence is immaterial. This is not the case in §6 when uncertainty in the output distribution is considered. We adopt the following conventions throughout the paper. The production quantity Q is finite. The terms increasing and decreasing are used in the weak sense.

We want to draw the reader's attention to certain assumptions. In our model, the firm specifies the product line and sells to a multitude of infinitesimal customers. As such, our model does not reflect all coproduct firms. For example, our model would be a poor fit for the semiconductor firm Cirrus because it produces custom chips, that is, the customer is heavily involved in specification, and its sales are dominated by one large customer; Apple accounted for 62% of Cirrus's total sales in fiscal year 2012. Our model reflects a firm that (i) produces noncustom products, often called catalog-type products in the semiconductor industry, and that (ii) sells to a broad customer base. There is evidence that such firms are relatively common in the semiconductor, LED, and industrial diamond industries.[5] Rather than explicitly modeling downstream entities as profit-maximizing firms, we adopt a utility function to model their purchasing behavior as it relates to quality. Although this utility approach is a simplification of reality, it does reflect the essential feature that downstream companies prefer higher quality to lower quality (at the same price). We also assume that customers do not further classify a grade purchased from the firm. Based on conversations with managers from semiconductor and industrial diamond firms, this is a reasonable assumption as additional classification by customers is not common due to technical and economic considerations.

4. Analysis

In this section we analyze the optimal pricing, product line (i.e., grade specification), and production decisions.

4.1. Pricing

We start by characterizing the customer purchasing behavior for a given grade specification vector $\mathbf{x}$.

[5] Many semiconductor firms, including Analog Devices, AMD, Freescale, Intel, and Texas Instruments (TI), are primarily catalog-type firms and/or have significant catalog-type business units within the company. Often, but certainly not always, the customer base of a catalog-type company is very large and not dominated by a few large customers. For example, according to Texas Instruments (2012), they "have more than 90,000 customers and, excluding our wireless baseband products, no single customer comprises more than 5% of our revenue." Analog Devices (2011) state that any one of their integrated-circuit products "can have as many as several hundred customers." In the case of LEDs, companies typically sell catalog-type products and not customized LEDs. Cree (2012) reported that no manufacturer accounted for more than 10% of its revenue. In the case of industrial diamonds, companies are privately held, but conversations with industry participants indicate that it is not uncommon for firms to have more than 50 customers.

Recall that N is the number of grades. Define $(x_0, p_0) = (0, 0)$ as the "outside option" that a customer obtains from not purchasing at all, and so each customer's outside utility is $\theta x_0 - p_0 = 0$. Confronted with the product line $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, a type θ customer chooses a grade by solving the problem $\max_{i=0,\ldots,N}\{\theta x_i - p_i\}$. As observed in other vertical-quality product line design papers, for example, Bhargava and Choudhary (2001) and Pan and Honhon (2012), there exists a set of indifference points $\{\theta_n\}$ with $\underline{\theta} = \theta_0 \le \theta_1 \le \cdots \le \theta_n \le \cdots \le \theta_N \le \theta_{N+1} = \bar{\theta}$ such that a customer with valuation $\theta \in [\theta_n, \theta_{n+1})$ has a first-choice preference for product (grade) n. These indifference points, or cutoffs, are given by $\theta_n = (p_n - p_{n-1})/(x_n - x_{n-1})$ for $n = 1, \ldots N$. A formal proof of these statements is given by Lemma A.1 in the appendix, §A.1. The first-choice demand for grade n is then given by $G(\theta_{n+1}) - G(\theta_n)$ for $n = 1, \ldots, N$.

We now turn our attention to the firm's grade-pricing decision. For any given production quantity Q and grade specification vector $\mathbf{x}$, the prices influence the firm's profit only through the revenue function $R(Q, \mathbf{x}, \mathbf{p})$. For a given price vector $\mathbf{p}$, the firm's revenue is $R(Q, \mathbf{x}, \mathbf{p}) = \sum_{n=1}^{N} p_n s_n(Q, \mathbf{x}, \mathbf{p})$, where $s_n(Q, \mathbf{x}, \mathbf{p})$ denotes the sales quantity of grade n. Letting $s_n^N(Q, \mathbf{x}, \mathbf{p}) = \sum_{i=n}^{N} s_n(Q, \mathbf{x}, \mathbf{p})$ denote the (echelon) sales quantity of grades $n, \ldots, N$, we can express the firm's revenue as $R(Q, \mathbf{x}, \mathbf{p}) = \sum_{n=1}^{N}(p_n - p_{n-1}) s_n^N(Q, \mathbf{x}, \mathbf{p})$, where we have used the fact that $p_0 = 0$ by definition. Because there is a one-to-one mapping between the cutoff vector $\boldsymbol{\Theta}$ and the price vector $\mathbf{p}$ given by $\theta_n = (p_n - p_{n-1})/(x_n - x_{n-1})$ for $n = 1, \ldots, N$, we can write

$$R(Q, \mathbf{x}, \boldsymbol{\Theta}) = \sum_{n=1}^{N} \theta_n (x_n - x_{n-1}) s_n^N(Q, \mathbf{x}, \boldsymbol{\Theta})$$

and optimize the revenue over the cutoff vector $\boldsymbol{\Theta}$ instead of the price vector $\mathbf{p}$.

It is instructive to first consider the case of a firm selling a single grade of quality x_1 for which it has infinite supply. For any given cutoff θ_1, the firm's revenue is $R(x_1, \theta_1) = x_1 \theta_1 \bar{G}(\theta_1)$, where $\bar{G}(\theta_1)$ is the customer demand at the price $x_1 \theta_1$. As proven in Lemma A.1 in the appendix, the optimal cutoff is given by θ^*, where θ^* is the unique θ that satisfies

$$\theta^* = \frac{\bar{G}(\theta^*)}{g(\theta^*)}.$$

In other words, $\bar{G}(\theta^*)$ is the revenue-maximizing quantity to sell in the infinite-supply single-grade case. We now present the optimal cutoffs (and hence optimal prices) for the general case. Recall that $Q_n^E = Q\bar{F}(x_n)$ is the echelon quantity of grade n, that is, the total quantity of grades $n, n+1, \ldots, N$.

Proposition 1. *For any given quantity Q and grade specification $\mathbf{x}$, the optimal cutoffs are given by $\theta_n^* = \max\{\theta^*, G^{-1}(1 - Q_n^E)\}$. Furthermore, the associated optimal revenue, denoted by $R(Q, \mathbf{x})$, is given by*

$$R(Q, \mathbf{x}) = x_{\hat{n}} \theta^* \bar{G}(\theta^*) + \sum_{n=\hat{n}+1}^{N} (x_n - x_{n-1}) G^{-1}(1 - Q_n^E) Q_n^E, \quad (1)$$

where $\hat{n}(Q, \mathbf{x})$ is the largest $n \le N$ such that $Q_n^E > \bar{G}(\theta^*)$ with $\hat{n}(Q, \mathbf{x}) = 0$ if $Q_1^E \le \bar{G}(\theta^*)$.[6]

To understand this proposition, let us first consider its implication when $N = 1$. In this single-grade case, the optimal cutoff is $\theta_1^* = \max\{\theta^*, G^{-1}(1 - Q_1^E)\}$. Noting that the firm sells $\bar{G}(\theta_1^*)$, this is equivalent to stating that the firm should price the grade so that demand equals supply if supply is limited, i.e., $Q_1^E < \bar{G}(\theta^*)$, but should price to sell the unconstrained revenue-maximizing quantity $\bar{G}(\theta^*)$ otherwise. When there is more than one grade, the optimal cutoff for grade n depends only on the echelon quantity $Q_n^E = Q\bar{F}(x_n)$. For those grades with a limited echelon supply, i.e., $Q_n^E < \bar{G}(\theta^*)$ or, equivalently, $n > \hat{n}$, the firm prices the grades so that the echelon demand (i.e., the quantity of customers wishing to purchase grade n or higher) matches the echelon supply. By backward recursion from N, it follows that the firm prices grades $n > \hat{n}$ so that demand for each grade exactly matches the supply of each grade. For grades $n \le \hat{n}$, the echelon supply is effectively unconstrained, and the firm prices so that the echelon sales of these lower grades equals the unconstrained revenue-maximizing quantity $\bar{G}(\theta^*)$. This means that the firm sets the same cutoff, θ^*, for these lower grades and, therefore, ensures no demand for grades $n < \hat{n}$. In effect, the firm "prunes" its product line to only sell grades $n \ge \hat{n}$, where, by definition, $\hat{n}$ is the highest grade whose echelon supply exceeds the revenue-maximizing quantity $\bar{G}(\theta^*)$. Because the echelon quantities increase in the production quantity Q, the firm prunes its product line more severely, that is, restricts sales to increasingly higher grades, as the production quantity increases for a given grade specification vector $\mathbf{x}$.

4.2. Product Line and Production Decisions

If the firm does not have classification technology, then the firm cannot classify its output and simply sells a single grade with a quality specification of $\underline{x}$, that is, the lower support of F. This "no-classification" case is characterized in §A.1. If the firm has classification technology, then it can sort the output into

[6] We use a convention that $\sum_{n=N+1}^{N} h_n(\cdot) = 0$ for any function $h_n(\cdot)$. Thus, $R(Q, \mathbf{x})$ is well-defined at $\hat{n}(Q, \mathbf{x}) = N$.

grades, and product line design, that is, choosing the number of grades N and the associated specification vector $\mathbf{x}$, becomes relevant. In all that follows, we assume the firm adopts and uses a classification technology unless otherwise stated. Of course, classification technology will be adopted if and only if it delivers a profit greater than the no-classification profit.

For a given Q, the product line design problem is to choose N and $\mathbf{x}$ to maximize $R(Q, \mathbf{x}) - C_B(Q, N)$, where the revenue $R(Q, \mathbf{x})$ is given by (1). The production cost $C_p(Q)$ does not depend on the product line at a given Q. The following proposition establishes two useful properties of an optimal product line.

Proposition 2. *For any given production quantity Q and number of grades N, an optimal specification vector must satisfy the following properties*:

(i) $x_1^*(Q) \geq x_{\min}(Q)$ *where* $x_{\min}(Q) = \underline{x}$ *if* $Q \leq \bar{G}(\theta^*)$ *and* $x_{\min}(Q) = F^{-1}[1 - \bar{G}(\theta^*)/Q]$ *otherwise*;

(ii) $x_{n+1}^*(Q) > x_n^*(Q)$ *for* $n = 1, \ldots N-1$.

We know from Proposition 1 that the firm prunes any grades whose echelon supply exceeds $\bar{G}(\theta^*)$, and these pruned grades do not influence the firm's revenue. Therefore, the firm does not benefit from grades whose echelon supplies exceed $\bar{G}(\theta^*)$, and so when designing its product line for a fixed Q and N, it should only select grades whose echelon supplies do not exceed $\bar{G}(\theta^*)$. This is formalized by property (i) above: $x_{\min}(Q)$ is the minimum quality level in the output range $[\underline{x}, \bar{x}]$ such that all higher qualities have echelon supplies no greater than $\bar{G}(\theta^*)$. Property (ii) states that (for any given Q and N) the firm will not create a degenerate grade such that $x_{n+1}^*(Q) = x_n^*(Q)$. Doing so would effectively reduce the number of grades by one, and therefore diminish the firm's ability to discriminate between customers (of different quality valuations) through its product line offering. We note that Propositions 1 and 2 together imply that there is positive supply and positive demand for every grade in an optimal product line (that is optimally priced), but more than that, they imply that the supply and demand exactly match for every grade.

Applying Propositions 1 and 2, we can write $R(Q, \mathbf{x})$ as

$$R(Q, \mathbf{x}) = \sum_{n=1}^{N} r(Q, x_{n-1}, x_n), \tag{2}$$

where

$$r(Q, x_{n-1}, x_n) = (x_n - x_{n-1}) G^{-1}(1 - Q\bar{F}(x_n)) Q\bar{F}(x_n). \tag{3}$$

Note that $r(Q, x_{n-1}, x_n)$ is the additional revenue obtained by adding a grade with specification $x_n > x_{n-1}$ to a preexisting grade specification vector $(x_0, x_1, \ldots, x_{n-1})$. Importantly, this incremental revenue depends on the preexisting grade vector only through the previous highest grade x_{n-1}. In other words, for a given production quantity Q, the revenue gain from adding a higher-quality product to an existing product line depends only on the previously highest-quality product and not on the entire product line.

This property allows us to formulate the product line optimization problem (for a given quantity Q) as a shortest-path network problem when the classification cost depends on the number of grades, i.e., $b_2 > 0$. See §A.2 for the shortest-path formulation.[7] We can therefore efficiently solve for the optimal $\mathbf{x}(Q)$. This algorithm needs to be run for each possible Q when solving for the optimal production quantity.[8] The optimal number of grades will decrease in b_2, and at a sufficiently high b_2 the firm will offer a single grade. This strategy, which we call a "separation strategy," is observed for certain powder products, for example, ultrafine Nickel powder, in which the firm remove particles above (or below) a certain size limit (JFE Mineral Company 2005). We analytically characterize the separation strategy, that is, the optimal grade quality and production quantity, in §A.1.

Although the classification cost will often depend on the number of grades, for example, size classification of powders by sieving requires a different sieve for each grade, there are situations in which the classification cost is (almost) independent of the number of grades, that is, $b_2 \approx 0$. For example, the operational cost of classifying semiconductors is dominated by the cost of testing each device, and this cost does not vary with the number of grades. We note that $b_2 = 0$ implies there are no costs to adding grades to a product line, and this assumption is made at times by many product line papers, for example, Moorthy (1984), Bhargava and Choudhary (2001), Netessine and Taylor (2007), Pan and Honhon (2012), and others. The following proposition proves that when $b_2 = 0$, the firm adopts a "complete-classification" strategy whereby it offers a grade at every quality point between the lowest and highest quality grades in the product line.[9]

[7] We are not the first to observe that shortest-path algorithms have application in product line design; Pan and Honhon (2012), for example, use a shortest-path algorithm to determine optimal product assortments in a uniproduct setting.

[8] Although we have not been able to establish that the profit at the optimal $\mathbf{x}^*(Q)$ is concave in Q, we can use a simple grid search over $0 \leq Q^* \leq Q_{\max}$, where $Q_{\max}$ is the Q such that $C_P(Q) + b_1 Q = \bar{x}\theta^* \bar{G}(\theta^*)$. This $Q_{\max}$ bound arises because $\bar{x}\theta^* \bar{G}(\theta^*)$ is an upper bound on the revenue for any Q, and so the profit is negative for $Q > Q_{\max}$.

[9] Even if the classification cost does not depend on the number of grades, there may be marketing and logistics costs that increase in

Proposition 3. *If $b_2 = 0$, then for any given Q,*
(i) *the optimal number of grades $N^*(Q) = \infty$;*
(ii) *the lowest grade is set as $x_1^*(Q) = x_{\min}(Q)$ and the highest grade is set as $x_N^*(Q) = \bar{x}$;*
(iii) *the resulting revenue $R(Q)$ is concave in Q and given by*

$$R(Q) = \begin{cases} Q\underline{x}G^{-1}(1-Q) + Q\int_{\underline{x}}^{\bar{x}} \bar{F}(x) \\ \quad \cdot G^{-1}(1-Q\bar{F}(x))\,dx, \quad Q \le \bar{G}(\theta^*), \\ F^{-1}\left(1 - \frac{\bar{G}(\theta^*)}{Q}\right)\theta^*\bar{G}(\theta^*) \\ \quad + Q\int_{F^{-1}(1-\bar{G}(\theta^*)/Q)}^{\bar{x}} \bar{F}(x) \\ \quad \cdot G^{-1}(1-Q\bar{F}(x))\,dx, \quad Q > \bar{G}(\theta^*); \end{cases} \tag{4}$$

(iv) *the profit $\Pi(Q) = R(Q) - Cp(Q) - C_B(Q, N^*(Q))$ is concave in Q.*

The firm sets its highest grade equal to the maximum quality it can produce. For low production quantities, it sets the lowest grade equal to the minimum quality produced, and the product line exactly matches the output spectrum $[\underline{x}, \bar{x}]$. At higher quantities, the firm benefits by discarding lower-quality output, and therefore sets the lowest grade to $x_1^*(Q) = F^{-1}[1 - \bar{G}(\theta^*)/Q]$; that is, the firm sets the lowest grade so that its echelon supply exactly matches the revenue-maximizing volume $G(\theta^*)$. Note that $x_1^*(Q)$ increases in Q because the firm is willing to discard more output as its production quantity increases. Whether the firm's optimal production quantity is low or not depends on the quantity-related production and classification costs. Part (iv) establishes that the production-quantity decision is well behaved. Closed-form expressions for the optimal product line, quantity, and profit when the customer-type and output distributions are both uniformly distributed are given in §A.4.

5. Process Technology

The "management of process technology is critical to firm strategy" in the semiconductor industry (Hatch and Mowery 1998, p. 1462) and in many other coproduct industries. A coproduct process can be characterized by its production cost function $C_P(Q)$ and its output distribution $F(\cdot)$. In this section we explore the impact of process technology on the firm's product line and profit.

5.1. Process Technology and Product Line

The existing product line design literature implicitly assumes that the firm operates a uniproduct technology. In that setting, the fundamental tension in product line design is that customers value higher quality (although to different degrees depending on their type), but the marginal production cost of a product depends on its quality. The first driver, that customers value quality, exists for vertical coproduct technologies, but the second does not because the marginal production cost is independent of the quality (grade) for coproduct technologies. However, the ability to produce a particular quality (grade) is constrained by the technology's output distribution. Therefore, in coproduct technologies, quality availability replaces the costliness of quality as a fundamental driver of product line design. This distinction leads to very different findings for coproduct and uniproduct technologies. Analogously to Netessine and Taylor (2007), we define the length of the product line as the difference in quality between highest and lowest grades offered.

Proposition 4. *In a coproduct technology with $C_p(Q) = cQ$,*
(i) *the optimal product line depends on the customer-type distribution $G(\cdot)$ even if $b_2 = 0$;*
(ii) *the optimal product line can contain multiple products and will never contain only the highest possible quality product;*
(iii) *the length of the optimal product line increases in c if $b_2 = 0$.*

These three results (in order) are in direct contradiction to the uniproduct findings of Pan and Honhon (2012, Corollary 4, p. 262), Bhargava and Choudhary (2001, Theorem 1, p. 96), and Netessine and Taylor (2001, Result 6, p. 112) described in §2. The reason for this difference lies in the fact that the output distribution, coupled with grade specification, creates an endogenous constraint on grade supply, a constraint that does not exist in uniproduct technologies. The product line never contains only the highest possible quality product because such a product line has infinitesimal supply in total.[10] To understand why the product line length increases in c, let us first consider how the production quantity Q influences product line length. The quality of the highest grade is constant in Q if $b_2 = 0$ (Proposition 3(ii)), and so the product line length increases if the quality of the lowest offered grade decreases. When Q is large, the firm has an ample supply of all qualities, including those at the high end of the output spectrum. It can, therefore, discard lower-quality output and sell only the higher end of the spectrum. When Q is small, however, the firm cannot afford to discard the lower end because it has a limited supply of higher-quality

the number of grades, and so we will not observe complete classification in practice because these other costs imply $b_2 > 0$. However, this $b_2 = 0$ case serves as a proxy for settings in which the fixed cost of adding grades is very low.

[10] If the output distribution is discrete rather than continuous as assumed, then this result may not hold.

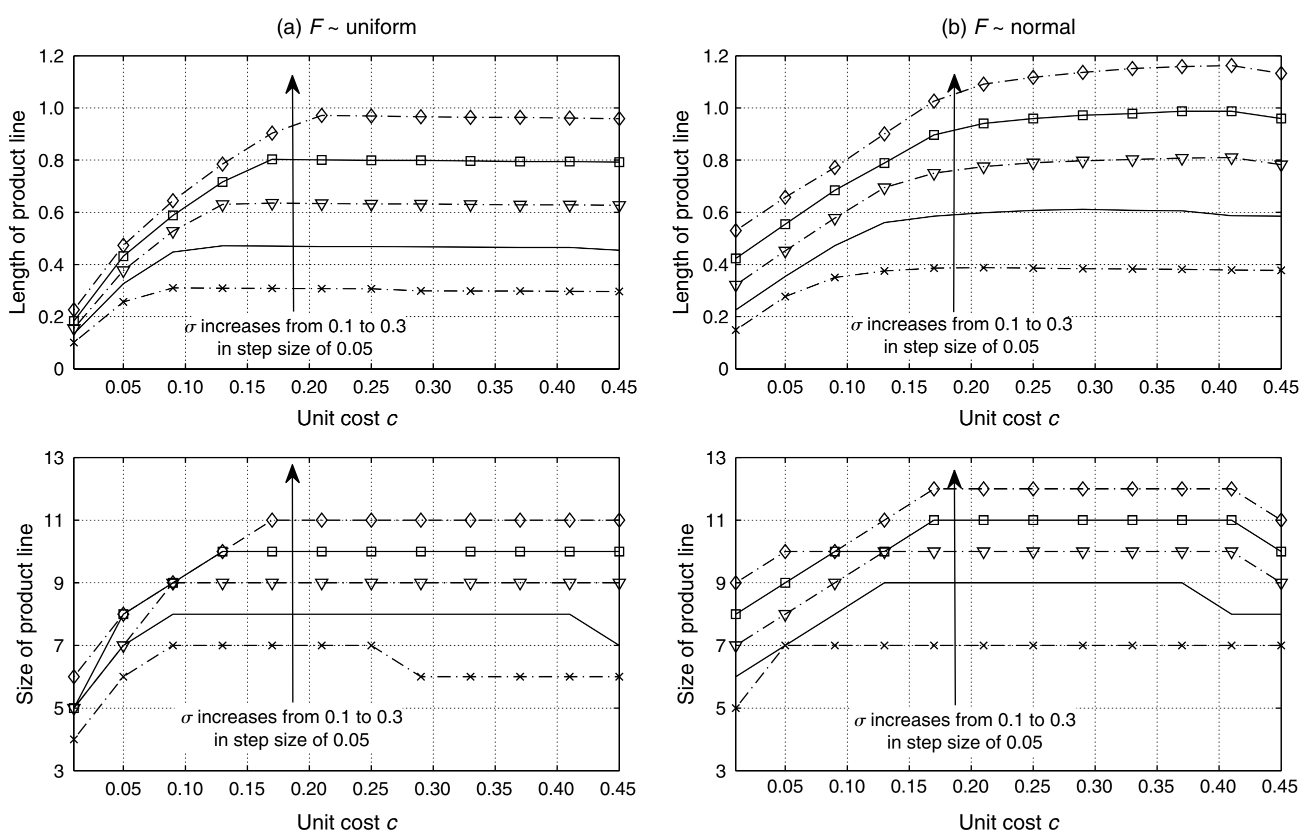
output. Thus, the quality of the lowest offered grade decreases as Q decreases. Equivalently, the product line length increases as Q decreases. The optimal production quantity decreases in the production cost c, and so the product line length increases in c.

We conducted a numerical study (available as supplemental material at http://dx.doi.org/10.1287/mnsc.2013.1738) to complement our analytical results. We varied the unit production cost c from 0.01 to 0.45 using a step size of 0.04. For the classification costs, we fixed $b_0 = 0$ but varied b_1 from 0%c to 10%c using a step size 2.5%c and varied b_2 from 0.001 to 0.005 using a step size 0.002. We fixed the mean of the output distribution $F(\cdot)$ as $\mu = 1.0$ but varied the standard deviation σ from 0.1 to 0.3 using a step size of 0.05 and also included $\sigma = 0.001$. For each parameter setting we considered both a uniform distribution and a normal distribution for $F(\cdot)$. We set the customer distribution as $G(\cdot) \sim \mathrm{U}(0, 1)$. This factorial design yielded 2,160 total instances. We also did a more limited study using a Beta distribution for $G(\cdot)$.

Echoing our earlier product line length result for $b_2 = 0$, that is, Proposition 4(iii), we observed numerically for $b_2 > 0$ that the product line length increased in the production cost c (unless c was very high). Different to the uniproduct technology finding of (Netessine and Taylor 2007, Result 1, p. 109), we also observed that the product line size, that is, the number of products/grades offered, increased in c (unless c was very high). The product line length and size observations are interrelated. By definition, there is a wide range in the offered output qualities when the product line length is large. Therefore, the firm wants to segment the offered output into many grades so as to maximize its revenues from a heterogeneous customer base; the fewer grades offered, the less able the firm is to discriminate between customers who value quality differently. When the product line length is small, however, there is not much range in the offered output and segmenting it into many grades is not very beneficial. It follows that the production cost c has a similar directional influence on both the product line length and size.

The effect of production cost on the product line length and size can be quite strong. Figure 1 presents the product line length and size as a function of the production cost c for different values of the standard deviation of the output distribution (with $b_1 = 0.05c$, $b_2 = 0.001$, and the output mean fixed at 1.0). Figure 1(a) uses a uniform distribution for $F(\cdot)$, and Figure 1(b) uses a normal distribution. The directional impact of the production cost and output variance are similar at higher values of b_2, but the product line length and size are both lower because the cost of offering more grades is higher.

Figure 1 Influence of Production Cost on Product Line Length and Size



We see that the product line size and length increase in the production cost c, but different from the case of $b_2 = 0$, the product line length can decease in c when c is very high. The same is true for the product line size. When there is a positive cost to offer a grade, i.e., $b_2 > 0$, the firm has to ensure a sufficiently large revenue to justify offering a grade. The firm only produces a small quantity Q when the production cost is high and so the supply of very high quality output is low. The firm reduces the quality of the highest-offered grade so that it has enough supply to generate sufficient revenue to merit offering the grade. Not surprisingly, Figure 1 also shows that the product line length and size both increase as the output distribution becomes more variable about a fixed mean. That the length increases can be proven when $b_2 = 0$ if $F(\cdot) \sim \mathrm{U}(a, b)$ and $G(\cdot) \sim \mathrm{U}(0, 1)$.

5.2. Process Technology and Profit: Process Innovation

Process innovation, whereby the firm seeks to improve the production technology, is a crucial aspect of operations strategy in many coproduct industries. We now examine what constitutes process improvement by exploring the impact of cost and output distribution on firm profit. In doing so, we consider four strategies: no classification, separation (i.e., single grade), complete classification, and optimal classification in which the firm can choose the number of grades. We use the labels *NC*, *S*, *CC*, and *OC* as shorthand for these four strategies, respectively.

Process development can reduce the production cost $C_P(\cdot)$ and/or alter the output distribution $F(\cdot)$. While a reduction in the cost is clearly an improvement (as the profit for each of the four strategies *NC*, *S*, *CC*, and *OC* decreases as $C_P(\cdot)$ increases), it is less clear what constitutes an improvement in the output distribution. Intuitively, if the output distribution shifts to the right on the quality spectrum, then this constitutes an improvement; formally if $F_2(\cdot)$ first-order stochastically dominates $F_1(\cdot)$, then the profit for each of the four strategies *NC*, *S*, *CC*, and *OC* is higher under $F_2(\cdot)$ than $F_1(\cdot)$ because for any given Q and $\mathbf{x}$, the echelon quantities of all grades are higher under $F_2(\cdot)$. Process innovation may not, however, lead to a first-order stochastically larger $F(\cdot)$. A new process may have the same mean but a lower variance. To investigate the impact of changes in variance at a fixed mean, we adopt the general notion of a mean-preserving spread (Machina and Pratt 1997), which includes as a special case an increase in variance if the allowed $F(\cdot)$ are restricted to a location–scale family (e.g., uniform and normal).

Proposition 5. *Let $F \uparrow_{\mathrm{MPS}}$ denote that $F(\cdot)$ becomes more variable in the mean-preserving spread sense. In a coproduct technology with $C_p(Q) = cQ$,*

(i) *Π^*_{NC} decreases as $F \uparrow_{\mathrm{MPS}}$;*

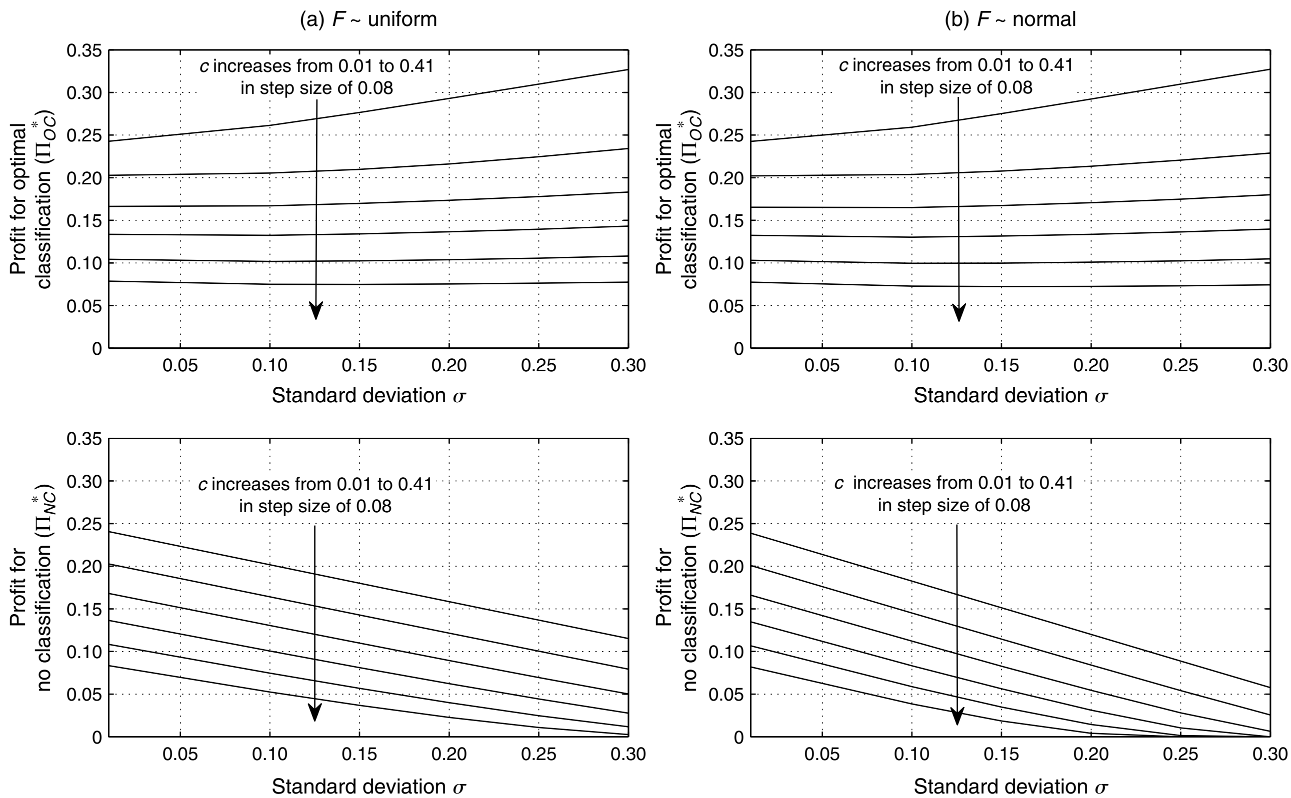
(ii) *there exists a threshold cost $\bar{c}$ such that Π^*_{OC} increases as $F \uparrow_{\mathrm{MPS}}$ for any $c \le \bar{c}$;*

(iii) *if $b_2 = 0$, Π^*_{OC} increases as $F \uparrow_{\mathrm{MPS}}$ for all c if $F(\cdot) \sim \mathrm{U}(a, b)$ and $G(\cdot) \sim \mathrm{U}(0, 1)$.*

If the firm lacks a classification technology, then a more variable output distribution (in the mean-preserving spread sense) always reduces profit because the quality of its product diminishes as $F(\cdot)$ becomes more variable. Therefore, variance reduction is a process improvement in the no-classification strategy. Remarkably, this is not the case when the firm adopts classification technology: variance amplification is a process improvement if the production cost c is low enough and is a process improvement for any c if the marginal classification cost $b_2 = 0$. The effect of a mean preserving spread can be quite strong. Figure 2 presents the optimal-classification profit and the no-classification profit as a function of the standard deviation of the output distribution (with a mean fixed at 1.0) for different values of the production cost c (with $b_1 = 0.05c$ and $b_2 = 0.001$). Figure 2(a) uses a uniform distribution for $F(\cdot)$, and Figure 2(b) uses a normal distribution.

There are two reasons that the optimal classification profit increases as $F \uparrow_{\mathrm{MPS}}$. The first reason lies with the impact of variance on the echelon supplies of the grades for a fixed grade specification $\mathbf{x}$. Suppose $F_1(\cdot)$ and $F_2(\cdot)$ exhibit the single-crossing property, i.e., $F_2(x) \ge F_1(x)\ \forall x < y$ and $F_2(x) \le F_1(x)\ \forall x \ge y$ for some y, which is true, for example, if $F_2(\cdot)$ is a mean-preserving spread of $F_1(\cdot)$ (Machina and Pratt 1997). Under the single-crossing property, the echelon supply $Q\bar{F}(x_n)$ under $F_2(\cdot)$ is higher (lower) than under $F_1(\cdot)$ for all n such that $x_n \ge y$ ($x_n < y$). Now, the optimal production quantity increases as the production cost c decreases, and thus the lower-bound $x_{\min}(Q^*)$ on x_1^* also increases. If c is low enough, then $x_1^* \ge y$ and so the echelon supplies of all grades increase as $F \uparrow_{\mathrm{MPS}}$, and this supply increase benefits the firm. The beneficial effect on echelon supplies is only part of the story. The firm can benefit as $F \uparrow_{\mathrm{MPS}}$ even when production costs are high, as reflected by Proposition 5(iii) and observed numerically for many instances with $b_2 > 0$. At high c, the optimal production quantity is low enough so that $x_n^* < y$ for some lower grades. The echelon supply of these lower grades decreases as $F \uparrow_{\mathrm{MPS}}$, but the detrimental effect of this reduction can be dominated by the beneficial effect of the increased echelon supplies of higher grades. Moreover, because the firm can tailor its grade vector specification $\mathbf{x}$ based on the output distribution, it can benefit as $F \uparrow_{\mathrm{MPS}}$ even if the detrimental effect would dominate at a fixed $\mathbf{x}$. A more variable output distribution enables the firm to create a product line with a larger separation in quality between

Figure 2 **Influence of Output Standard Deviation on Profit**



the highest and lowest grades, and reminiscent of the product line literature, for example, Deneckere and McAfee (1996), this benefits the firm because it can better discriminate among the heterogeneous quality-valuation customer population. As already observed in §5, the product line length (and size) increases as the output variance increases.

The firm's operations strategy becomes more sophisticated as it moves from no classification to separation to optimal classification, and Proposition 5 sheds light on the value of classification, which we define as $V_C = \Pi^*_{OC} - \Pi^*_{NC}$. When $b_2 = 0$, Proposition 5 implies that V_C increases as $F \uparrow_{\mathrm{MPS}}$ (when $F(\cdot) \sim \mathrm{U}(a, b)$ and $G(\cdot) \sim \mathrm{U}(0, 1)$). Numerically we observed that V_C also increases as $F \uparrow_{\mathrm{MPS}}$ for $b_2 > 0$. Classification allows the firm to extract value from higher-quality units rather than selling them as low-quality units, and this benefit increases as the spread between high and low qualities in the output distribution increases. Intuitively, then, one might also expect that the value of multigrading, defined as $V_M = \Pi^*_{OC} - \Pi^*_S$, that is, the value of being able to offer multiple grades instead of a single grade, should also increase as $F \uparrow_{\mathrm{MPS}}$. We did observe this in our numerical study. So, classification, whether in its simplest form (separation) or its more sophisticated form (multiple grades), is particularly important for process technologies with highly variable output distributions.

Although the profits of all strategies decrease in the production cost c, they do not necessarily decrease at the same rate. When $b_2 = 0$, the value of classification V_C decreases in c when $F(\cdot) \sim \mathrm{U}(a, b)$ and $G(\cdot) \sim \mathrm{U}(0, 1)$ (proof omitted). Numerically we observed that V_C also decreases in c for $b_2 > 0$. Interestingly, we observed that the value of multigrading V_M increases in c. In other words, offering multiple grades is particularly important for process technologies with high production costs. That multigrading is more beneficial as c increases follows from the fact discussed above that the length and size of the optimal product line both increase in c as discarding low-quality output becomes more expensive.

6. Uncertain Output Distribution

We now show how our results extend to the case where the output distribution is uncertain. In particular, the output distribution is $F_s(\cdot)$ in scenario $s = 1, \ldots, S$, with the probability of scenario s being ξ_s. The production-quantity decision is made before the scenario is revealed, with the objective of maximizing the expected profit. The product line and prices might be set before ("advance") or after ("recourse") the scenario is revealed. We assume recourse pricing and examine advance and recourse product line design. We first note that for a given quantity Q and product line $\mathbf{x}$, the optimal recourse prices (cutoffs) and associated revenue $R_s(Q, \mathbf{x})$ are given by Proposition 1, but with the output distribution being given by the realized $F_s(\cdot)$, and so echelon supplies are scenario dependent.

If $\mathbf{x}$ is set in recourse, then all our product line design results from §4 and §A.1 continue to hold, but now apply at whichever $F_s(\cdot)$ has occurred. Because concavity is preserved under expectation with respect to the scenario s, the expected profits for complete classification, no classification, and separation continue to be concave in the production quantity. The analysis is more complicated when $\mathbf{x}$ is set in advance, but all earlier product line and quantity results can be extended to the advance case, with some appropriate modifications. We have developed counterparts to all the propositions in this paper for the advance case, and they can be found (along with their proofs) in an unabridged appendix available from the authors.

As with the deterministic output case, under recourse product line design the firm never prunes any of the grades it chooses to offer, that is, it never prices an offered grade so that no customer wants it as its first choice. This is not necessarily the case with advance product line design. There will be at least one scenario in which the firm will not prune any of the grades but there may be other scenarios in which it does; the reason being that the echelon supply of grades are scenario dependent but grades cannot be adapted to the scenario in the advance case.

Recourse product line design will dominate advance product line design because the firm can tailor its product line to the realized output distribution. Interestingly, when $b_2 = 0$, the expected profits are the same under both advance and recourse settings (proof in the unabridged appendix). Complete classification, that is, offering an infinite number of grades, is optimal when $b_2 = 0$. In this case, advance design results in a product line that contains within its interval all the scenario-dependent ones under recourse design. Therefore, infinite grading along with recourse pricing enables the advance line $\mathbf{x}$ to capture the same revenue in any scenario s as the recourse line $\mathbf{x}_s$. This expected profit equivalency result is not generally true for $b_2 > 0$.

In the uncertain output distribution case, a process technology is defined by the production cost $C_p(Q)$ and the set of possible distributions $\{F_s(\cdot)\}$. The impact of $C_p(Q)$ on the product line (Proposition 4) continues to hold under both recourse and advance product line design. With regard to the impact of the output distribution (Proposition 5), when we say the output becomes more variable in the uncertain output case we mean that $F_s(\cdot)$ becomes more variable in the mean-preserving spread sense for all s, or more generally, some $F_s(\cdot)$ become more variable while the others remain unchanged. With this interpretation, Proposition 5 continues to hold under both recourse and advance product line design; that is, the no-classification expected profit decreases, but the optimal classification expected profit can increase as the output becomes more variable.

7. Conclusion

Coproducts are an essential attribute of the process technology in many industries. In this paper we analyzed the product line design and production decisions of a firm that operates a coproduct technology in which the output differs in quality in the vertical sense. We characterized the optimal prices, product line, and production quantity. Different from uniproduct technology where the firm can produce products independently, coproduct technology influences product line design not because of the cost of quality but because the output distribution constrains the firm's ability to supply quality levels. This fundamental distinction leads to differences between the two technology types with regard to the influence of the customer-type distribution and production costs on the optimal product line. For example, the size and length of the product line both increase in the production cost for coproduct technologies.

Process innovation is an important aspect of operations strategy in coproduct industries. We examined how a coproduct technology, characterized by its production cost function and output distribution, influences the firm's profit. We formally established the intuitive notion that a first-order stochastically larger output distribution is a process improvement. More surprisingly, perhaps, we proved that variability amplification (in the mean-preserving spread sense) is a process improvement if the production or classification costs are low enough. We showed that the capability to classify into multiple grades (rather than separating into one grade) is particularly important if production costs are high.

Our model represents a firm that sells to a large number of small customers. This reflects many practical settings but certainly not all. There are many cases in which a coproduct firm has a small number of dominant customers. It would be interesting to examine product line design for this alternative setting by treating the firm's direct customers as profit-maximizing entities who cater to their downstream consumers.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/mnsc.2013.1738.

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Appendix

A.1. The No-Classification and Separation Strategies

If the firm has no classification technology, then it offers a single grade with a quality specification of $\underline{x}$, that is, the lower support of F.

Proposition A.1. *For the no-classification strategy,*

(i) $\Pi(Q)$ *is concave in* Q;

(ii) *if* $C'_P(0) < \underline{x}$, *then* Q^* *is given by the solution to* $G^{-1}(1-Q) - Q/g(G^{-1}(1-Q)) = C'_P(Q)/\underline{x}$, *and* $Q^* = 0$ *otherwise;*

(iii) *the optimal profit is given by* $\Pi^* = (\underline{x}Q^{*2})/(G^{-1}(1-Q^*))$ *if* $C_P(Q) = cQ$.

Closed-form expressions for the optimal quantity and profit under no classification when the customer types are uniformly distributed are given by Corollary A.2 in §A.4.

In the separation strategy the firm offers a single grade but has classification technology and so can set the grade's quality specification.

Proposition A.2. *For the separation strategy, if F is an IFR distribution, then*

(i) *the revenue* $R(x_1, Q)$ *is unimodal in* x_1, *and* $x_1^*(Q) = \underline{x}$ *if* $J(Q, \underline{x}) < 0$ *and otherwise* $x_1^*(Q)$ *is the unique* x_1 *that satisfies* $J(Q, x_1) = 0$, *where*

$$J(Q, x_1) = Q\bar{F}(x_1) + g(G^{-1}(1 - Q\bar{F}(x_1))) \cdot G^{-1}(1 - Q\bar{F}(x_1))\left(\frac{\bar{F}(x_1)}{x_1 f(x_1)} - 1\right); \tag{5}$$

(ii) *the profit* $\Pi(Q) = R(x_1^*(Q), Q) - C_P(Q) - C_B(Q, N = 1)$ *is concave in* Q.

Similar to the complete-classification strategy, $x_1^*(Q)$ is increasing in Q because the firm is willing to discard more output as its production quantity increases; see Corollary A.1 in §A.3.

A.2. Product Line Design Algorithm

We can formulate the product line optimization problem (for a given quantity Q) as a shortest-path network problem if we represent the quality interval $(x_{\min}(Q), \bar{x})$ as $T+1$ discrete points and restrict potential grade specifications to these quality points. See Steps 1–4 below. Because shortest-path problems can be solved efficiently, we can use very large values of T to effectively eliminate any precision loss resulting from the discretization of the quality interval $(x_{\min}(Q), \bar{x})$. The algorithm needs to be run for each possible Q when solving for the optimal production quantity.

Step 1. Create a network with $T+2$ nodes, labeled $i = 0, \ldots, T+1$, with a directed arc (i, j) from every node $i = 0, \ldots, T$ to every node $j > i$.

Step 2. Assign the following $x(i)$ values to each node: $x(0) = 0$, $x(i) = x_{\min}(Q) + (i-1)((\bar{x} - x_{\min}(Q))/T)$ for $i = 1, \ldots, T+1$.

Step 3. Assign the following costs to each arc (i, j):

$$c(i, j) = \begin{cases} -r(Q, x(i), x(j)), & i = 0; \\ -r(Q, x(i), x(j)) + b_2, & i > 0. \end{cases}$$

Step 4. Compute the shortest path from node 0 to node $T+1$. The nodes in the shortest path, or more precisely, the associated $x(i)$ values, correspond to the optimal grade specification vector at the given Q. The optimal profit is the absolute value of the length of the shortest path.

A.3. Proofs

Proofs have been compressed (some significantly) for reasons of space. Detailed proofs are contained in an unabridged appendix available from the authors. Lemmas and their proofs are given in §A.1.

A.3.1. Propositions.

Proof of Proposition 1. We have $R(Q, \mathbf{x}, \mathbf{\Theta}) = \sum_{n=1}^{N} \theta_n (x_n - x_{n-1}) s_n^N(Q, \mathbf{x}, \mathbf{\Theta})$, where $s_n^N(Q, \mathbf{x}, \mathbf{\Theta})$ are the echelon sales of grade n. The echelon inventory is $Q_n^E = Q_n + \cdots + Q_N$. Let D_n denote the first-choice demand for grade n. Using Lemma A.1, $D_n = G(\theta_{n+1}) - G(\theta_n)$, and $D_N = \bar{G}(\theta_N)$. Therefore, the echelon first-choice demand is $D_n^N = D_n + \cdots + D_N = \bar{G}(\theta_n)$. Under the assumption that the firm downward substitutes (if needed) and that unfilled customers spill down, the sales of grades $n, \ldots, N$ is the minimum of echelon inventory and the echelon first-choice demand, i.e., $s_n^N(Q, \mathbf{p}, \mathbf{x}) = \min\{Q_n^E, \bar{G}(\theta_n)\}$. We then have

$$R(Q, \mathbf{x}, \mathbf{\Theta}) = \sum_{n=1}^{N} \theta_n (x_n - x_{n-1}) \min\{Q_n^E, \bar{G}(\theta_n)\}. \tag{6}$$

The firm optimizes the cutoff vector $\mathbf{\Theta}$ subject to $\theta_n \geq \theta_{n-1}$ for $n = 1, \ldots, N$. We first ignore this ordering constraint but prove that the optimal cutoffs to the unconstrained problem conform to this ordering. Observe that (6) is separable in the θ_n. We can determine the optimal θ_n by maximizing $\theta_n \min\{Q_n^E, \bar{G}(\theta_n)\}$ because $x_n \geq x_{n-1}$ by definition. Because $\bar{G}(\theta)$ decreases in θ and because $\theta_n \bar{G}(\theta_n)$ is unimodal in θ_n with a maximum attained at $\theta_n^* = \theta^*$ (see Lemma A.2), it follows that $\theta_n^* = \max\{\theta^*, G^{-1}(1 - Q_n^E)\}$. We now show that the optimal cutoffs conform to $\theta_n^* \geq \theta_{n-1}^*$: $1 - Q_n^E$ is increasing in n because the echelon inventory Q_n^E is decreasing in n. Thus, $\theta_n^* = \max\{\theta^*, G^{-1}(1 - Q_n^E)\}$ is increasing in n. The echelon quantity Q_n^E is decreasing in n. Define $\hat{n}(Q, \mathbf{x})$ to be the largest $0 \leq n \leq N$ such that $Q_n^E > \bar{G}(\theta^*)$ with $\hat{n}(Q, \mathbf{x}) = 0$ if $Q\bar{F}(x_1) \leq \bar{G}(\theta^*)$. It follows that $Q_n^E > \bar{G}(\theta^*)$ for all $n \leq \hat{n}(Q, \mathbf{x})$, and $Q_n^E \leq \bar{G}(\theta^*)$ for all $n > \hat{n}(Q, \mathbf{x})$. Therefore, $\theta_n^* = \theta^*$ for all $n \leq \hat{n}(Q, \mathbf{x})$, and $\theta_n^* = G^{-1}(1 - Q_n^E)$ for all $n > \hat{n}(Q, \mathbf{x})$. Then, (1) follows from (6) after some algebra. This completes the proof. □

Note that Proposition 1 is true even if spill-down does not occur and/or if downward substitution is not used. Call the problem with spill-down/substition P1, and the problem without (or with limited) spill-down/substition P2. Both problems can be set up as a nonlinear problem with echelon sales as a decision variable but including constraints (quantity and demand) on the echelon sales. P1 is a relaxed version of P2. The optimal solution to P1 (given by Proposition 1) is in fact feasible for P2 because Proposition 1 implies that at the optimal solution (i) first-choice demand equals grade inventory for grades $n > \hat{n}$, (ii) first-choice demand is less than grade inventory for grade $\hat{n}$, and (iii) there is no first-choice demand for grades $n = 1, \ldots, \hat{n} - 1$.[11] Therefore,

[11] Using Lemma A.1, $D_n = G(\theta_{n+1}) - G(\theta_n)$. Now $\theta_n^* = G^{-1}(1 - Q_n^E)$ for all $n > \hat{n}$. Thus, $G(\theta_n^*) = 1 - Q_n^E$ and $D_n = 1 - Q_{n+1}^E - (1 - Q_n^E) = Q_n^E - Q_{n+1}^E = Q_n$; that is, first-choice demand for bucket n is exactly equal to the inventory of bucket n for all $n > \hat{n}$. Next, consider $n = \hat{n}$ for which $\theta_{\hat{n}}^* = \theta^*$ because $\theta_n^* = \theta^*$ for $n \leq \hat{n}$. Thus, $D_{\hat{n}} = 1 - Q_{\hat{n}+1}^E - G(\theta^*)$.

there is no spill-down/substition at the optimal prices (cutoffs), and so the optimal solution to the relaxed problem P1 is feasible (and hence optimal) for P2.

Spill-down means that a customer (whose order is not filled) spills to her next-preferred *lower* grade until her order is filled or no inventory of lower grades is available. In this paper we do not allow spill-up whereby a customer spills to a higher grade if her next preferred product happens to be a higher grade. Although spill-up does not occur at the optimal solution given in Proposition 1, this does not mean the solution is optimal for a model with spill-up; the reason being that P1 is not a relaxed version of a problem that allows spill-up.

Proof of Proposition 2. The classification cost $C_B(Q, N)$ depends on $\mathbf{x}$ only through N. Thus, for any given Q and N, maximizing $R(Q,\mathbf{x}) - C_B(Q,N)$ over $\mathbf{x}$ is equivalent to maximizing $R(Q,\mathbf{x})$ over $\mathbf{x}$. We first prove (ii) and then (i).

(ii) Let $\mathbf{x}$ be such that $\hat{n}(Q,\mathbf{x}) = N$. Using (1), we have $R(Q,\mathbf{x}) = x_N\theta^*\bar{G}(\theta^*)$ because $\sum_{n=\hat{N}+1}^{N} = 0$ by convention. Therefore, $R(Q,\mathbf{x})$ is strictly increasing in x_N, and so $\mathbf{x}$ cannot be optimal. Next, let $\mathbf{x}$ be such that $0 < \hat{n}(Q,\mathbf{x}) < N$. Using (1), we have $R(Q,\mathbf{x}) = x_{\hat{n}}\theta^*\bar{G}(\theta^*) + \sum_{n=\hat{n}+1}^{N}(x_n - x_{n-1})Q_n^E G^{-1}\cdot(1-Q_n^E)$. Taking the partial derivative with respect to (w.r.t.) $x_{\hat{n}}$ we have $\partial R(Q,\mathbf{x})/\partial x_{\hat{n}} = \theta^*\bar{G}(\theta^*) - Q_{\hat{n}+1}^E G^{-1}(1-Q_{\hat{n}+1}^E)$. Now, $Q_{\hat{n}+1}^E G^{-1}(1-Q_{\hat{n}+1}^E) = \hat{\theta}\bar{G}(\hat{\theta})$, where $\hat{\theta} = G^{-1}(1-Q_{\hat{n}+1}^E)$. Therefore, $\partial R(Q,\mathbf{x})/\partial x_{\hat{n}} = \theta^*\bar{G}(\theta^*) - \hat{\theta}\bar{G}(\hat{\theta})$. Now, from Lemma A.2, $\theta\bar{G}(\theta)$ is unimodal in θ with its maximum value at θ^*. Therefore, $\partial R(Q,\mathbf{x})/\partial x_{\hat{n}} > 0$ if $\hat{\theta} > \theta^*$. By definition of $\hat{n}(Q,\mathbf{x})$, $Q_n^E > \bar{G}(\theta^*)$ for all $n \geq \hat{n}$. Therefore, $\hat{\theta} = G^{-1}(1 - Q_{\hat{n}+1}^E) > \theta^*$. This proves $\partial R(Q,\mathbf{x})/\partial x_{\hat{n}} > 0$ for any $\mathbf{x}$ such that $0 < \hat{n}(Q,\mathbf{x}) < N$. It then follows that an optimal $\mathbf{x}$ must have $\hat{n}(Q,\mathbf{x}) = 0$. Consider an arbitrary $\mathbf{x}$ with $\hat{n}(Q,\mathbf{x}) = 0$. Using (1), we can write $R(Q,\mathbf{x}) = \sum_{n=1}^{N} r(Q, x_{n-1}, x_n)$, where $r(Q, x_{n-1}, x_n) = (x_n - x_{n-1})G^{-1}(1 - Q\bar{F}(x_n))Q\bar{F}(x_n)$. Let $\mathbf{x}$ have the property that $x_{\check{n}+1} = x_{\check{n}}$ for some $\check{n} = 1, \ldots, N-1$. Construct a new $\mathbf{x}$, denoted by $\mathbf{x}_k$, which is identical to $\mathbf{x}$ except $x_{\check{n}+1} = x_k$, where $x_{\check{n}} < x_k < x_{\check{n}+2}$. Then, $R(Q,\mathbf{x}_k) - R(Q,\mathbf{x}) = r(Q, x_{\check{n}}, x_k) + r(Q, x_k, x_{\check{n}+2}) - r(Q, x_{\check{n}}, x_{\check{n}+1}) - r(Q, x_{\check{n}+1}, x_{\check{n}+2})$. Now, $r(Q, x_{\check{n}+1}, x_{\check{n}}) = 0$ because $x_{\check{n}+1} = x_{\check{n}}$. Therefore, $R(Q,\mathbf{x}_k) - R(Q,\mathbf{x}) = r(Q, x_{\check{n}}, x_k) + r(Q, x_k, x_{\check{n}+2}) - r(Q, x_{\check{n}}, x_{\check{n}+2})$. Applying Lemma A.4, we then have $R(Q,\mathbf{x}_k) > R(Q,\mathbf{x})$. (Note that we can use Lemma A.4 because $Q\bar{F}(x_{\check{n}}) \leq Q\bar{F}(x_1) \leq \bar{G}(\theta^*)$ as $\hat{n}(Q,\mathbf{x}) = 0$.) Therefore, $\mathbf{x}$ cannot be optimal. This proves that $x_{n+1}^*(Q) > x_n^*(Q)$ for $n = 1, \ldots, N-1$.

(i) By definition $x_{\min}(Q) = \underline{x}$ if $Q \leq \bar{G}(\theta^*)$ and $x_{\min}(Q) = F^{-1}[1 - \bar{G}(\theta^*)/Q]$ otherwise. We have already proven that $\hat{n}(Q,\mathbf{x}^*) = 0$. Therefore, by definition of $\hat{n}(Q,\mathbf{x})$, we have $Q\bar{F}(x_1^*(Q)) \leq \bar{G}(\theta^*)$. If $Q > \bar{G}(\theta^*)$, then $x_1^*(Q) \geq F^{-1}[1 - \bar{G}(\theta^*)/Q] = x_{\min}(Q)$. If $Q \leq \bar{G}(\theta^*)$, then $x_{\min}(Q) = \underline{x}$. Recall that $x_0 = 0$ by definition. For any $x_1(Q) < \underline{x}$, we have $r(Q, x_1, x_0) = x_1 G^{-1}(1-Q)Q$ because $\bar{F}(x_1(Q)) = \bar{F}(\underline{x}) = 1$. Now $r(Q, x_1, x_0)$ is strictly increasing in x_1 in this region and so $x_1^*(Q) \geq \underline{x} = x_{\min}(Q)$, which completes the proof. □

Therefore, $D_{\hat{n}} \leq Q_n \Leftrightarrow \bar{G}(\theta^*) \leq Q_n + Q_{\hat{n}+1}^E \Leftrightarrow \bar{G}(\theta^*) \leq Q_{\hat{n}}^E$, which is true by definition of $\hat{n}$. In other words, there is sufficient quantity of bucket $\hat{n}$ to fill first-choice demand for bucket $\hat{n}$. We next consider buckets $n \leq \hat{n} - 1$. For these buckets $\theta_n^* = \theta^*$ and so $D_n = G(\theta^*) - G(\theta^*) = 0$ and there is no first-choice demand for these buckets.

Proof of Proposition 3. (i) When $b_2 = 0$, $C_B(Q,N) = b_0 + b_1 Q$ is independent of N and $\hat{\mathbf{x}}$. Therefore, the firm selects an N and $\hat{\mathbf{x}}$ to maximize $R(Q,\mathbf{x})$. It follows from Proposition 2 and its proof that an optimal $\mathbf{x}$ must have $\hat{n}(Q,\mathbf{x}) = 0$. Thus, using (1), we can write $R(Q,\mathbf{x}) = \sum_{n=1}^{N} r(Q, x_{n-1}, x_n)$, where $r(Q, x_{n-1}, x_n) = (x_n - x_{n-1})G^{-1}(1 - Q\bar{F}(x_n))Q\bar{F}(x_n)$. Let $\mathbf{x}^*(N,Q)$ denote the optimal specification vector if the firm uses N grades. Construct a new $\mathbf{x}$ with $N+1$ grades by splitting grade n into two grades so that this new grade vector, denoted by $\mathbf{x}_k(N+1,Q)$, is $x_1^* < \cdots < x_{n-1}^* < x_n^* < x_k < x_{n+1}^* < \cdots < x_N^*$; that is, x_k is the grade introduced. Then,

$$
\begin{aligned}
&R(Q, \mathbf{x}_k(N+1,Q)) - R(Q, \mathbf{x}^*(N,Q)) \\
&\quad = r(Q, x_n^*, x_k) + r(Q, x_k, x_{n+1}^*) - r(Q, x_n^*, x_{n+1}^*),
\end{aligned}
$$

and applying Lemma A.4, we then have $R(Q,\mathbf{x}_k(N+1,Q)) > R(Q,\mathbf{x}^*(N,Q))$. (Note that we can use Lemma A.4 because $Q\bar{F}(x_n^*) \leq Q\bar{F}(x_1^*) \leq \bar{G}(\theta^*)$ as $\hat{n}(Q,\mathbf{x}^*) = 0$.) We have proven that there exists a feasible grade specification for $N+1$ grades with a strictly greater revenue than for the optimal specification for the N grade case. It follows that the optimal revenue $R(Q,\mathbf{x}^*(N,Q))$ is strictly increasing in N, which proves part (i).

(ii) Tailoring $R(Q,\mathbf{x})$ to the case of $N = \infty$, we obtain

$$
\begin{aligned}
R(Q, x_1, \hat{x}) = Q\bigg(& x_1\bar{F}(x_1)G^{-1}(1 - Q\bar{F}(x_1)) \\
&+ \int_{x_1}^{\hat{x}} \bar{F}(x)G^{-1}(1 - Q\bar{F}(x))\,dx\bigg),
\end{aligned}
$$

where $\hat{x}$ denotes the highest grade. Now, $\partial R(Q, x_1, \hat{x})/\partial\hat{x} = Q\bar{F}(\hat{x})G^{-1}(1 - Q\bar{F}(\hat{x})) \geq 0$, and so $\hat{x}^* = \bar{x}$. We then have

$$
\begin{aligned}
R(Q, x_1) = Q\bigg(& x_1\bar{F}(x_1)G^{-1}(1 - Q\bar{F}(x_1)) \\
&+ \int_{x_1}^{\bar{x}} \bar{F}(x)G^{-1}(1 - Q\bar{F}(x))\,dx\bigg). \qquad (7)
\end{aligned}
$$

Taking the derivative w.r.t. x_1,

$$
\begin{aligned}
\frac{\partial R(Q, x_1)}{\partial x_1} = Qx_1 f(x_1)\bigg(& -G^{-1}(1 - Q\bar{F}(x_1)) \\
&+ \frac{Q\bar{F}(x_1)}{g(G^{-1}(1 - Q\bar{F}(x_1)))}\bigg).
\end{aligned}
$$

It can be shown (see the unabridged appendix) that $\partial R(Q, x_1)/\partial x_1 \leq 0$ because $\theta_1^* \geq \theta^*$ using Proposition 2(i). It is therefore optimal to set x_1 to the minimum possible value such that $\theta_1^* = G^{-1}(1 - Q\bar{F}(x_1)) \geq \theta^*$, which implies $x_1^*(Q) = x_{\min}(Q)$, where $x_{\min}(Q) = \underline{x}$ if $Q \leq \bar{G}(\theta^*)$ and $x_{\min}(Q) = F^{-1}[1 - \bar{G}(\theta^*)/Q]$ otherwise.

(iii) $\underline{x} \leq F^{-1}[1 - \bar{G}(\theta^*)/Q] \Leftrightarrow Q \leq \bar{G}(\theta^*)$. Substituting x_1^* into (7) yields the expression in $R(Q)$ expression. Please see the unabridged appendix for proof that $R(Q)$ is concave.

(iv) $\Pi(Q) = R(Q) - C_P(Q) - C_B(Q, N^*(Q))$. Now, $C_P(Q)$ is convex in Q. Also, $C_B(Q, N^*(Q)) = b_0 + b_1 Q$ when $b_2 = 0$, and so linear in Q. Therefore, $\Pi(Q) = R(Q) - C_P(Q) - C_B(Q, N^*(Q))$ is concave because $R(Q)$ is concave in Q from part (iii). □

Proof of Proposition 4. (i) Proof follows from Proposition 3 in which case $x_1^*(Q) = x_{\min}(Q)$, where $x_{\min}(Q) = \underline{x}$ if $Q \le \bar{G}(\theta^*)$ and $x_{\min}(Q) = F^{-1}[1 - \bar{G}(\theta^*)/Q]$ otherwise. Clearly $x_1^*(Q)$ and hence x_1^* is not independent of $G(\cdot)$. This proof was based on $b_2 = 0$ and so x_1^* is not independent of $G(\cdot)$ in general. Even if $b_2 > 0$, the fact that x_1^* is not independent of $G(\cdot)$ can be established using Proposition A.2 for example.

(ii) That the optimal product line can contain multiple products (and will contain infinite products if $b_2 = 0$) follows directly from Proposition 3. If the firm selects a single grade, then, using Proposition A.2, the optimal grade (for a given Q) is $x_1^*(Q) = \underline{x}$ if $J(Q, \underline{x}) < 0$, and otherwise $x_1^*(Q)$ is the unique x_1 that satisfies $J(Q, x_1) = 0$, where $J(Q, x_1)$ is given by (5). Now, at $x_1 = \bar{x}$, $J(Q, \bar{x}) = -g(G^{-1}(1))G^{-1}(1) < 0$, and so $x_1^*(Q) < \bar{x}$ for all Q and so $x_1^* < \bar{x}$.

(iii) By definition, the product line length is $x_N^* - x_1^*$, where N is the number of grades in the optimal grade specification vector $\mathbf{x}^*$. When $b_2 = 0$, the lowest grade is set as $x_1^*(Q) = x_{\min}(Q)$, and the highest grade is set as $x_N^*(Q) = \bar{x}$. Therefore, the product line length is $\bar{x} - x_{\min}(Q)$ at any given Q. Now, $x_{\min}(Q)$ increases in Q, and so $x_1^* = x_{\min}(Q^*)$ decreases in c as Q^* decreases in c. Therefore, the optimal product line length $\bar{x} - x_1^*$ increases in c. □

Proof of Proposition 5. (i) The lower support $\underline{x}$ of $F(\cdot)$ (weakly) decreases as $F \uparrow_{\text{MPS}}$. Observe from the proof of Proposition A.1 that $\Pi'_{NC}(Q)$ decreases in $\underline{x}$ for all Q. It follows that $\Pi_{NC}(Q^*)$ decreases in $\underline{x}$, and so Π_{NC}^* (weakly) decreases as $F \uparrow_{\text{MPS}}$.

(ii) Let $F_2(\cdot)$ be a mean-preserving spread of $F_1(\cdot)$, and let y denote the single-crossing point (Machina and Pratt 1997) such that $F_2(x) \ge F_1(x)\ \forall\, x < y$ and $F_2(x) \le F_1(x)\ \forall\, x \ge y$. Let Q^* and $\mathbf{x}^*$ denote the optimal production quantity and specification vector under $F_1(\cdot)$. The quantity Q^* decreases in c, and so $x_{\min}(Q^*)$ decreases in c. Now, $x_1^* \ge x_{\min}(Q^*)$ (Proposition 2), and so there exists a threshold cost $\bar{c}$ such that $x_1^* \ge y$ for all $c \le \bar{c}$. Therefore, $\bar{F}_2(x_n^*) \ge \bar{F}_1(x_n^*)$ for all n if $c \le \bar{c}$, and so the echelon quantities $Q\bar{F}_2(x_n^*) \ge Q\bar{F}_1(x_n^*)$ for all n if $c \le \bar{c}$. Recall from (2) that $R(Q, \mathbf{x}) = \sum_{n=1}^{N} r(Q, x_{n-1}, x_n)$, where $r(Q, x_{n-1}, x_n) = (x_n - x_{n-1})G^{-1}(1 - Q\bar{F}(x_n))Q\bar{F}(x_n)$ or, equivalently, $R(Q, \mathbf{x}) = \sum_{n=1}^{N} r(Q_n^E, x_{n-1}, x_n)$, where $r(Q, x_{n-1}, x_n) = (x_n - x_{n-1})G^{-1}(1 - Q_n^E)Q_n^E$ and $Q_n^E = Q\bar{F}(x_n)$. Now, $r(Q_n^E, x_{n-1}, x_n)$ is increasing in Q_n^E (Lemma A.5), and so $R(Q^*, \mathbf{x}^*)$ is higher under $F_2(\cdot)$ than $F_1(\cdot)$ if $c \le \bar{c}$. Now, as defined above, Q^* and $\mathbf{x}^*$ are optimal under $F_1(\cdot)$, and so the profit $\Pi(Q^*, \mathbf{x}^*) = R(Q, \mathbf{x}^*) - C(Q^*) - B(Q^*, N^*)$ under $F_2(\cdot)$ is at least as large as the optimal profit under $F_1(\cdot)$ if $c \le \bar{c}$. (Recall that N^* is the number of grades in $\mathbf{x}^*$.)

(iii) Complete classification is optimal for $b_2 = 0$ (Proposition 3), and the optimal profit Π^* is given in Corollary A.3 for $F(\cdot) \sim \mathrm{U}(a, b)$ and $G(\cdot) \sim \mathrm{U}(0, 1)$. If $F(\cdot) \sim \mathrm{U}(a, b)$, then $F \uparrow_{\text{MPS}}$ is equivalent to σ increasing at a constant μ, where $\mu = (a + b)/2$ and $\sigma = (b - a)/(2\sqrt{3})$ are the mean and standard deviation of F. Proof then follows by taking derivative of Π^* with respect to σ and verifying the derivative is positive (details omitted for reasons of space). □

Proof of Proposition A.1. (i) Because $x_1 = \underline{x}$, the echelon inventory $Q_1^1 = Q$, and so $\hat{n} = 0$ if $Q < \bar{G}(\theta^*)$ and $\hat{n} = 1$ otherwise. Tailoring Proposition 1 to this no-classification case we obtain $R(Q) = \underline{x}QG^{-1}(1 - Q)$ if $Q < \bar{G}(\theta^*)$ and $R(Q) = \underline{x}\theta^*\bar{G}(\theta^*)$ if $Q \ge \bar{G}(\theta^*)$. The firm's profit is $\Pi(Q) = R(Q) - cQ$, and this is continuous and differentiable (at the boundary $Q = \bar{G}(\theta^*)$). Using Lemma A.3, $QG^{-1}(1 - Q)$ is concave in Q, and so $\Pi(Q)$ is concave in Q.

(ii) The first derivative of the profit function is $\Pi'(Q) = \underline{x}(G^{-1}(1 - Q) - Q/g(G^{-1}(1 - Q))) - C'_P(Q)$ if $Q < \bar{G}(\theta^*)$ and $\Pi'(Q) = -C'_P(Q)$ if $Q \ge \bar{G}(\theta^*)$. Noting that $\Pi'(Q) = \underline{x} - C'_P(0)$ at $Q = 0$ and $\Pi'(Q) = -C'_P(Q) < 0$ for $Q \ge \bar{G}(\theta^*)$, it follows from part (i) that Q^* is given by the solution to the first-order condition, i.e., $G^{-1}(1 - Q) - Q/g(G^{-1}(1 - Q)) = C'_P(Q)/\underline{x}$ if $C'_P(0) < \underline{x}$ but $Q^* = 0$ otherwise.

(iii) $Q^* < \bar{G}(\theta^*)$ and $\underline{x}G^{-1}(1 - Q^*) = \underline{x}Q^*/g(G^{-1}(1 - Q^*)) + c$ from proof of (ii) when $C_P(Q) = cQ$. The profit $\Pi(Q) = \underline{x}QG^{-1}(1 - Q) - cQ$ for $Q < \bar{G}(\theta^*)$ from proof of part (i). Proof follows by substitution. □

Proof of Proposition A.2. (i) The classification cost $C_B(Q, N) = b_0 + b_1 Q + b_2$ when the firm pursues a separation strategy, and so the product line design x_1 does not influence $C_B(Q, N)$. Therefore, the firm selects an x_1 to maximize $R(Q, x_1)$. Adapting Proposition 1 to the single-grade problem, we have

$$R(Q, x_1) = G^{-1}(1 - Q\bar{F}(x_1))x_1 Q\bar{F}(x_1), \tag{8}$$

$$\frac{\partial R(Q, x_1)}{\partial x_1} = \begin{cases} G^{-1}(1-Q)Q, & x_1 < \underline{x}, \\ Q\biggl(\dfrac{x_1 f(x_1)Q\bar{F}(x_1)}{g(G^{-1}(1 - Q\bar{F}(x_1)))} + G^{-1}(1 - Q\bar{F}(x_1)) \\ \qquad \cdot (\bar{F}(x_1) - x_1 f(x_1))\biggr), & \underline{x} \le x_1 \le \bar{x}. \end{cases} \tag{9}$$

Recall, from Proposition 2 that $x_1^*(Q) \ge x_{\min}(Q)$, where $x_{\min}(Q) = \underline{x}$ if $Q \le \bar{G}(\theta^*)$ and $x_{\min}(Q) = F^{-1}[1 - \bar{G}(\theta^*)/Q]$ otherwise. We prove that $R(Q, x_1)$ is either monotonic or unimodal in x_1 over $x_1 \ge x_{\min}(Q)$ by proving it is unimodal over a wider range, namely, that it unimodal over $x_1 \ge \hat{x}_{\min}(Q)$ where $\hat{x}_{\min}(Q) \le x_{\min}(Q)$. In particular, we define $\hat{x}_{\min}(Q) = x_{\min}(Q)$ if $Q > \bar{G}(\theta^*)$ and $\hat{x}_{\min}(Q) = 0 \le \underline{x}$ if $Q < \bar{G}(\theta^*)$. Proof that $R(Q, x_1)$ is unimodal in x_1 for $x_1 \ge \hat{x}_{\min}(Q)$ follows by application of Lemma A.4 in the e-companion of Aydin and Porteus (2008), namely, that if $f(x)$ is a twice continuously differentiable function of $x \ge \hat{x}_{\min}$, then $f(x)$ is unimodal if (a) $f(x)$ is strictly increasing in x at $x = \hat{x}_{\min}$, (b) $f(x)$ is strictly decreasing in x as x tends to ∞, and (c) $f''(x) < 0$ at any x that satisfies $f''(x) = 0$. In particular, we prove in the unabridged appendix that, at any given Q, (a) $R(Q, x_1)$ is strictly increasing in x_1 at $x_1 = \hat{x}_{\min}(Q)$, (b) $R(Q, x_1)$ is strictly decreasing in x_1 as x_1 tends to its limit $\bar{x}$, and (c) $\partial^2 R(Q, x_1)/\partial x_1^2 < 0$ at any x_1 that satisfies $\partial R(Q, x_1)/\partial x_1 = 0$. The rest of the proposition statement follows by rearranging (9) (when $\underline{x} \le x \le \bar{x}$) to obtain $J(Q, x_1)$.

(ii) Using (8), the envelope theorem, and (9) we have (see the unabridged appendix for details) $\partial\Pi(Q)/\partial Q = -C'_P(Q) - b_1 + G^{-1}(1 - Q\bar{F}(x_1^*))(\bar{F}(x_1^*)^2/f(x_1^*))$, where we have dropped the dependence of x_1^* on Q for notational ease. Let $R'(Q) = G^{-1}(1 - Q\bar{F}(x_1^*))(\bar{F}(x_1^*)^2/f(x_1^*))$. To prove that $\Pi(Q)$ is concave in Q, we only need to show that $R''(Q) \le 0$ because $C_P(Q)$ is convex in Q. It can be shown (see the unabridged appendix for details) that $R''(Q) < -(\partial x_1^*/\partial Q)\bar{F}(x_1^*)\hat{\theta}^*(1 + (\bar{F}(x_1^*)/f(x_1^*))(1/x_1^* + f'(x_1^*)/f(x_1^*)))$. Now, because $\partial x_1^*/\partial Q \ge 0$ (Corollary A.1), it follows

$R''(Q) \le 0$ is $1+(\bar{F}(x_1^*)/f(x_1^*))\{1/x_1^*+f'(x_1^*)/f(x_1^*)\} \ge 0$. Rearranging terms, this is equivalent to $x(f^2(x_1^*)+f'(x_1^*)\bar{F}(x_1^*))+\bar{F}(x_1^*)f(x_1^*) \ge 0$, which is equivalent to $F(\cdot)$ being an increasing generalized failure rate (IGFR) distribution and is therefore satisfied if $F(\cdot)$ is an IFR distribution. This can also be directly observed because $f^2(x_1^*)+f'(x_1^*)\bar{F}(x_1^*) \ge 0$ if $F(\cdot)$ is an IFR distribution. Therefore, $R''(Q) \le 0$, and so $\Pi(Q)$ is concave in Q. □

Corollary A.1. *x_1^* increases in Q.*

Proof of Corollary A.1. The proof is contained in the unabridged appendix available from authors.

A.3.2. Technical Lemmas.

Property A.1. *If the customer-type distribution $G(\cdot)$ is an IFR distribution, then $G(\cdot)$ satisfies $2g^2(\theta)+\bar{G}(\theta)g'(\theta) \ge 0$.*

Proof of Property A.1. The failure rate function is $v(\theta) = g(\theta)/\bar{G}(\theta)$ and $v'(\theta) = (\bar{G}(\theta)g'(\theta) + g^2(\theta))/\bar{G}(\theta)^2$. By definition, $G(\cdot)$ is an IFR distribution if and only if (iff) $v'(\theta) \ge 0$. Therefore, $\bar{G}(\theta)g'(\theta) + g^2(\theta) \ge 0$. It follows that $2g^2(\theta)+\bar{G}(\theta)g'(\theta) \ge 0$ because $g(\cdot) \ge 0$. □

Lemma A.1. *For any* $\mathbf{x}$, (a) *there exists a set of cutoffs $\{\theta_n\}$ with $\underline{\theta} = \theta_0 \le \theta_1 \le \theta_2 \le \cdots \le \theta_n \le \cdots \le \theta_N \le \theta_{N+1} = \bar{\theta}$ such that a customer with valuation $\theta \in [\theta_n, \theta_{n+1})$ has a first-choice preference for grade n.* (b) *Furthermore, there exists an optimal price vector such that the set of cutoffs $\{\theta_n\}$ is given by $\theta_n = \theta_{n-1}^n$ for $n = 1, \ldots, N$.*

Proof of Lemma A.1. Confronted with the product line $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, a type θ customer chooses a grade by solving the problem $\max_{i=0,\ldots,N}\{\theta x_i - p_i\}$. This corresponds to the customers' incentive compatibility constraint. Incidentally, this also includes the individual rationality constraints that the resulting utility should be nonnegative, because $\max_{i=0,\ldots,N}\ \{\theta x_i - p_i\} \ge 0$ as $(x_0, p_0) = (0, 0)$. Tie breaking when choosing a grade is arbitrary because customers are continuously distributed in an interval; thus, the measure of indifferent customers is zero. (a) Consider a pair (i, j) where $i < j$, and suppose that there exists a type θ who weakly prefers (x_j, p_j) to (x_i, p_i), i.e., $x_j > x_i$ and $\theta x_j - p_j \ge \theta x_i - p_i$. Now consider an arbitrary type $\tilde{\theta} > \theta$. We obtain that $\tilde{\theta}x_j - p_j = (\tilde{\theta}-\theta)x_j + \theta x_j - p_j \ge (\tilde{\theta}-\theta)x_j + \theta x_i - p_i = (\tilde{\theta}-\theta)x_j - \tilde{\theta}x_i + \theta x_i + \tilde{\theta}x_i - p_i = (\tilde{\theta}-\theta)(x_j - x_i) + \tilde{\theta}x_i - p_i > \tilde{\theta}x_i - p_i$, where the first inequality follows from the construction of type θ, and the second inequality follows from $\tilde{\theta} > \theta$ and $x_j > x_i$. Thus, a type $\tilde{\theta}$ customer strictly prefers (x_j, p_j) to (x_i, p_i). As a mirror image, if there exists a type $\hat{\theta}$ who weakly prefers (x_i, p_i) to (x_j, p_j), we can show that all types below $\hat{\theta}$ strictly prefer (x_i, p_i) to (x_j, p_j). Collectively, if none of these two grades dominates the other, then there must exist a customer who is indifferent between choosing either grade. We label this type as θ_i^j. By definition, the following condition must be satisfied: $\theta_i^j x_j - p_j \ge \theta_i^j x_i - p_i \Leftrightarrow \theta_i^j = (p_j - p_i)/(x_j - x_i)$. All types above θ_i^j strictly prefer (x_j, p_j) to (x_i, p_i), whereas all types below θ_i^j strictly prefer (x_i, p_i) to (x_j, p_j). This also suggests that it is impossible to find a set of customers whose first choice is a lower-quality product than some set of customers with lower valuations. Therefore, the customer valuations must separate into sets such that the first-choice grade of customers in a higher set is a higher grade than the first-choice grade of customers in lower set. Labeling "grade" 0 as customers who do not purchase, we then have the ordering specified in the statement. (b) Consider the case in which the sets $[\theta_n, \theta_{n+1})$ are all nonempty. If $\theta_n < \theta_{n-1}^n$, then there are customers in $[\theta_n, \theta_{n+1})$ that prefer $n-1$ to n, which is a contradiction. If $\theta_n > \theta_{n-1}^n$, then there exist customers in $[\theta_{n-1}, \theta_n)$ that prefer n to $n-1$, which is also a contradiction. Therefore, we must have $\theta_n = \theta_{n-1}^n$. Note that part (a) does not guarantee that all sets are nonempty. However, if there exist some empty sets, then the prices of the associated grades can be decreased without any loss of revenue and still retain the original sets but ensuring the cutoffs are given by $\theta_n = \theta_{n-1}^n$ for $n = 1, \ldots, N$. □

Lemma A.2. (a) *$\theta\bar{G}(\theta)$ is unimodal in θ.* (b) *A unique θ^* exists that maximizes $\theta\bar{G}(\theta)$, and θ^* satisfies $\theta^* = \bar{G}(\theta^*)/g(\theta^*)$.*

Proof of Lemma A.2. (a) If $G(\cdot)$ is an IFR distribution, then $G(\cdot)$ is IGFR (Lariviere 2006). That $\theta\bar{G}(\theta)$ is unimodal in θ follows from (Lariviere 2006, p. 602; with θ replacing p in the $\pi(p)$ expression). (b) Let $r(\theta) = \theta\bar{G}(\theta)$. Proof follows by setting $r'(\theta)$ equal to 0. □

Lemma A.3. (a) *$H(Q) = QG^{-1}(1-Q)$ is concave in Q.* (b) *$H(Q)$ is maximized at $Q^* = \bar{G}(\theta^*)$.*

Proof of Lemma A.3. (a) Taking the second derivative of $H(Q)$, we have $H''(Q) \le 0 \Leftrightarrow 2(g(G^{-1}(1-Q)))^2 + Qg'(G^{-1}(1-Q)) \ge 0$. Letting $\theta = G^{-1}(1-Q)$ (which implies $Q = \bar{G}(\theta)$), it follows that $H''(Q) \le 0$ iff $2(g(\theta))^2 + \bar{G}(\theta)g'(\theta) \ge 0$, which is true by Property A.1. Therefore, $H(Q) = QG^{-1}(1-Q)$ is concave in Q. (b) At $Q = \bar{G}(\theta^*)$, $H'(Q) = G^{-1}(G(\theta^*)) - \bar{G}(\theta^*)/g(G^{-1}(G(\theta^*)) = \theta^* - \bar{G}(\theta^*)/g(\theta^*)) = 0$, where the final equality follows from Lemma A.2. □

Lemma A.4. *For $z_1 < z_2$ and $Q\bar{F}(z_1) \le \bar{G}(\theta^*)$, define $r(Q, z_1, z_2) = (z_2 - z_1)G^{-1}(1 - Q\bar{F}(z_2))Q\bar{F}(z_2)$. Then $r(Q, z_1, z_2) < r(Q, z_1, z_s) + r(Q, z_s, z_2)$ for $z_1 < z_s < z_2$.*

Proof of Lemma A.4. By definition, $r(Q, z_1, z_s) + r(Q, z_s, z_2) - r(Q, z_1, z_2) = (z_s - z_1)(G^{-1}(1 - Q\bar{F}(z_s))Q\bar{F}(z_s) - G^{-1}(1 - Q\bar{F}(z_2))Q\bar{F}(z_2))$. Now, $z_s > z_1$, and so $r(Q, z_1, z_2) < r(Q, z_1, z_s) + r(Q, z_s, z_2)$ if $G^{-1}(1 - Q\bar{F}(z_s))Q\bar{F}(z_s) > G^{-1}(1 - Q\bar{F}(z_2))Q\bar{F}(z_2)$. Defining $\theta_2 = G^{-1}(1 - Q\bar{F}(z_2))$ and $\theta_s = G^{-1}(1 - Q\bar{F}(z_s))$, it follows that $r(Q, z_1, z_2) < r(Q, z_1, z_s) + r(Q, z_s, z_2)$ if $\theta_2\bar{G}(\theta_2) > \theta_s\bar{G}(\theta_s)$. Now, from Lemma A.2, $\theta\bar{G}(\theta)$ is unimodal in θ with its maximum value at θ^*. Therefore, $r(Q, z_1, z_2) < r(Q, z_1, z_s) + r(Q, z_s, z_2)$ if $\theta_2 > \theta_s > \theta^*$. Now, $Q\bar{F}(z_1) \le \bar{G}(\theta^*)$ and $z_1 < z_s < z_2$ from the lemma statement. Therefore, $\theta_2 > \theta_s > \theta^*$. □

Lemma A.5. *$r(Q_n^E, x_{n-1}, x_n)$ is increasing in $Q_n^E = Q\bar{F}(x_n)$ where*

$$r(Q_n^E, x_{n-1}, x_n) = \begin{cases} (x_n - x_{n-1})\theta^*\bar{G}(\theta^*), & Q_n^E \ge \bar{G}(\theta^*); \\ (x_n - x_{n-1})G^{-1}(1 - Q_n^E)Q_n^E, & Q_n^E < \bar{G}(\theta^*). \end{cases}$$

Proof of Lemma A.5. The incremental revenue $r(Q_n^E, x_n, x_{n-1})$ is constant in Q_n^E for $Q_n^E \ge \bar{G}(\theta^*)$. For $Q_n^E > \bar{G}(\theta^*)$,

$r(Q_n^E, x_n, x_{n-1}) = (x_n - x_{n-1})\theta(Q_n^E)\bar{G}(\theta(Q_n^E))$, where $\theta(Q_n^E) = G^{-1}(1 - Q_n^E)$. Now, $\theta(Q_n^E) > \theta^*$ as $Q_n^E \geq \bar{G}(\theta^*)$. Furthermore, $\theta(Q_n^E)$ is decreasing in Q_n^E. From Lemma A.2, $\theta\bar{G}(\theta)$ is decreasing in θ for $\theta > \theta^*$. It then follows that $r(Q_n^E, x_n, x_{n-1})$ is increasing in Q_n^E because $x_n > x_{n-1}$. □

A.4. Expressions for Uniformly Distributed Customer Types

Proofs of the following corollaries can be found in the unabridged appendix. We note that $\mu = (a+b)/2$ and $\sigma = (b-a)/(2\sqrt{3})$ are the mean and standard deviation of $F(\cdot) \sim \mathrm{U}(a, b)$, respectively. Also, $\hat{c} = c + b_1$.

Corollary A.2. *For the no-classification strategy, if $C_P(Q) = cQ$ and $G(\cdot) \sim \mathrm{U}(0, 1)$, then*

(i) $Q^* = (\underline{x} - c)/(2\underline{x})$ *and* $\Pi^* = (\underline{x} - c)^2/(4\underline{x})$ *if* $c < \underline{x}$, *and* $Q^* = 0$ *and* $\Pi^* = 0$ *otherwise;*

(ii) *if* $F(\cdot) \sim \mathrm{U}(a, b)$, *then* $Q^* = [\mu - \sigma\sqrt{3} - c]^+/(2(\mu - \sigma\sqrt{3}))$ *and* $\Pi^* = ([\mu - \sigma\sqrt{3} - c]^+)^2/(4(\mu - \sigma\sqrt{3}))$.

Corollary A.3. *If $b_2 = 0$, $C_P(Q) = cQ$, and if $G(\cdot) \sim \mathrm{U}(0, 1)$ and $F(\cdot) \sim \mathrm{U}(a, b)$, then*

$$Q^* = \begin{cases} \dfrac{1}{2(3^{1/4})}\sqrt{\dfrac{\sigma}{\hat{c}}}, & 0 < \hat{c} < \dfrac{\sigma}{\sqrt{3}}; \\[2ex] \dfrac{[\mu - \hat{c}]^+}{2(\mu - \sigma/\sqrt{3})}, & \hat{c} \geq \dfrac{\sigma}{\sqrt{3}}; \end{cases}$$

$$x_1^* = \begin{cases} \mu + \dfrac{\sigma^2 - \sigma\sqrt{3}(\mu + \hat{c})}{\mu - \hat{c}}, & 0 < \hat{c} < \dfrac{\sigma}{\sqrt{3}}; \\[2ex] \mu - \sigma\sqrt{3}, & \hat{c} \geq \dfrac{\sigma}{\sqrt{3}}; \end{cases}$$

$$x_N^* = \mu + \sqrt{3}\sigma;$$

$$\Pi^* = \begin{cases} \dfrac{1}{4}\left(\mu + \sqrt{3}\sigma - 4\left(\dfrac{1}{3}\right)^{1/4}\sqrt{\hat{c}\sigma}\right) - b_0, & 0 < \hat{c} < \dfrac{\sigma}{\sqrt{3}}; \\[2ex] \dfrac{([\mu - \hat{c}]^+)^2}{4(\mu - \sigma/\sqrt{3})} - b_0, & \hat{c} \geq \dfrac{\sigma}{\sqrt{3}}. \end{cases}$$

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